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# *THE NATURAL HISTORY OF THE VIBRATO*

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The vibrato has been studied in the psychological laboratory of the University of Iowa from two approaches: (1) phonophotographic recording of musical performance and speech, and (2) psychophysic measurements on the perception of the vibrato produced synthetically by instruments. The following may be regarded as samples of facts that have been established by experiment.*

A general outcome is definition of the phenomenon. An artistic vibrato consists of a periodic oscillation in pitch in which the extent of oscillation for the best singers averages approximately a half-tone and for string instruments approximately a quarter-tone, at an average rate of approximately six or seven cycles per second, and is usually accompanied by synchronous intensity and timbre oscillations which play a secondary rôle.

The following qualitative definition is recommended for musical terminology: A good vibrato in music is a periodic pulsation, generally involving pitch, intensity, and timbre, which produces a pleasing flexibility, mellowness and richness of tone.

The constituent factors being known, a quantitative definition may, of course, be formulated for countless purposes: e.g., the parallel vibrato is one in which the periodic rise of the pitch and the increase in intensity are synchronous and in phase.

The vibrato is present in the voices of all great artists in about 95 per cent of their phonated time, including transition tones, attacks, releases, and rapid passages, as well as sustained tones. The same is approximately true of primitive peoples, such as Indians and Negroes in so far as they sing with great feeling in their native milieu.

It is occasionally found in early childhood and its occurrence increases with age among those who sing with feeling. It is also occasionally present in speech, especially in the sustained vowels of emotional and dramatic speech.

At the present time it is used on sustained tones in nearly all string

instruments. No great violinist of today plays without a vibrato. Occasionally it is also used with wind instruments.

As implied in the definition, the constituent factors are pitch, intensity, and timbre as registered against time determining rate. The final character of a vibrato tone is determined quantitatively in terms of the contribution of each of these variables to the blend in the tone progression. Pitch, intensity, and timbre each vary in extent, rate, and form.

By extent we mean the distance between the top and bottom of the crest, expressed in fractions of a tone. The average extent of the pitch cycle in the singing of artists of today is about six-tenths of a tone, roughly a semi-tone. The same is true of primitive people and adult vocal students but children have a smaller extent.

The average variability in pitch extent for individual singers is quite uniform. Fifty percent of the extents in pitch lie between 0.4 and 0.8 of a tone with a sharp peak between 0.5 and 0.7. The extremes run from 0.1 to 1.0+. The extent of the pitch vibrato in voice does not vary to any marked degree with the actual loudness of the tone, the pitch register, the tone placement, the vowel quality, the sex of the singer, or the musical modes.

The rate of the pitch vibrato, that is, the number of pulsations per second, averages between six and seven cycles. Fifty per cent of the artists' rates fall between six and seven. There is no marked relationship between rate of pitch oscillation and loudness, tone placement, vowel quality, register, sex, or musical mode.

The form of the pitch vibrato approximates that of the sine wave. Minor irregularities and tendencies to distortion occur, but these are probably of but slight musical significance.

Turning to the characteristics of the intensity factor, we find that the average extent of the intensity pulsation in the song of artists is 2.4 decibels. There being about 9 or 10 perceptible differences in the extent of the pitch oscillation, this means that the intensity oscillation is only about one-fourth as perceptible as the pitch oscillation, each taken by itself.

The curve of variability of intensity extents takes about the same form as for pitch extents. The rate of the intensity vibrato is, as a rule, synchronous with the pitch vibrato although not always in phase.

The third variable is timbre. By slowing down a phonograph record one can hear marked changes in timbre of the tone from trough to crest in the vibrato cycle. Harmonic analysis shows these changes to be exceedingly complex so that, although they are conspicuous, it has not yet been possible to formulate any definite rules for the variation in timbre of the tone.

The relation of pitch to intensity presents three basic forms: forty per cent of vibratos are "parallel," that is, the high pitch coincides with the

greater intensity; about 30 per cent are "opposite," that is, there is a reversed relationship; in about 50 per cent, though synchronous, there is a partial shift in phase. The intensity vibrato is present in about 70 per cent of all vocal pitch vibratos.

The most astonishing finding in the whole investigation is the fact that, although the vibrato is almost universally regarded as a medium of differential expression of emotion, measurements show that vocal vibrato does not differentiate to any significant degree between musical modes, between types of emotion, or even between extreme calmness and excitement. This is a shocking finding because the whole practice of the vibrato at the present time rests largely on the assumption that it is a means for differential expression of the feeling of the artist. The cold fact is, however, that the singing vibrato is like an organ stop. You have it in or you have it out and while the performer secures a considerable range of variety in pitch, intensity, timbre, and rate, these are not differentiated. The situation is different in stringed instruments when the player voluntarily changes the vibrato in accordance with what he thinks it should be. Yet even here there is a remarkable uniformity in continuous playing.

Experiments show that it is comparatively easy to develop a vibrato in a child or adult who sings without it, it is possible to administer remedial training both in extent and rate of the different factors, and it is possible to set up artistic goals or norms which may be reached by appropriate training.

As an example of the significance of this, we may say that while the average vibrato of the best singers on the stage today is about a half-tone, experiments tend to show that probably the most pleasing vibrato on the average would be that of the violin, namely, a quarter of a tone or slightly less. If this conception should be recognized, it would be a comparatively easy matter to reduce the vibrato of the best singers of today by about one-half and thereby create a radically new situation for the musical critic, actors and performers.

We do not yet have a complete solution of the physiological explanation of the vibrato. It is clear, however, that physiologically there are several forms of vibrato; that the vibrato is one of the basic neuro-muscular periodicities observable not only in the voice but in various parts of the body, not only in man but also in animals; that in the vibrato there are higher periodicities of two or three orders, riding on the vibrato wave; and that the vibrato is allied to the tremor.

This objective description shows the vibrato to be a gross distortion of tone far greater than is heard in listening to music and it is this underestimation or failure to perceive the vibrato as such which accounts for the tolerance and the utter confusion in regard to the nature and rôle of the

vibrato in the entire history of the subject. It is also an essential condition.

Among the numerous normal illusions which function in the vibrato, four are outstanding. One is the astonishing under-estimation of the magnitude of the vibrato in hearing, which lies in the fact that the oscillation of pitch is heard as if it were only one-fourth to one-half of its actual extent and a similar under-estimation occurs for intensity.

Another normal illusion consists in the persistent confusion of oscillations in pitch and intensity in listening so that a musician, or even a generation of musicians, will assert that it is oscillation in pitch, that it is oscillation in intensity, or, more frequently, that it is neither but some unknown form of pulsation in tone quality.

A third normal illusion, which makes the vibrato in its present gross form tolerable, is the phenomenon of sonance, which lies in the fact that successive periodicities, when of sufficient rate, tend to fuse into a unified tone somewhat in the same manner that the simultaneous overtones in a violin clang fuse and are heard together as one tone.

A fourth normal illusion, which is a condition for making the vibrato tolerable, is the fact that even with a pitch oscillation of a semi-tone the intonation is heard as of a particular tone which can easily be identified with an even or true pitch, the muscial effect heard being that of a changing tone quality. Without these four and numerous similar normal illusions which function in all musical hearing, the vibrato as it now exists would be utterly intolerable.

Thus as in all art, illusion plays a vital rôle in the situation. In this case, the perception of the vibrato as beautiful rests upon one's being subject to these illusions. After all, normal illusions are among the sweetest things in life.

In general, the vibrato in speech is like the vibrato in song in all its characteristics. In their origin, instrumental vibratos are all imitations of vocal vibrato.

In studying the vibrato we have registered song, speech, and instrumental music in such a way as to collect the material for similar treatment of other factors in music and speech from the point of view of deviation from the regular or exact in tonal, dynamic, temporal, or qualitative characteristics of sounds.

I call this the natural history of the vibrato because what we have done is to go out as the botanist does and collect living material for the purpose of classification and experimental studies.

* A full account of these studies appears in "*The Vibrato*," Univ. of Iowa Studies in the Psychology of Music now in press.

# MEASURES OF CERTAIN HUMAN ABILITIES THROUGHOUT THE LIFE SPAN

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Children, youths and adults of college age have been psychologically measured and studied a great deal. Men and women in middle age and especially those in later maturity have largely escaped such investigation; they were not readily available in our laboratories and because of the strong interest that workers have had in the early period of development few have attempted to obtain their coöperation. Now we are beginning to realize that so long as we lack scientific data on the actual changes in human abilities throughout the life span, just so long will we continue merely to guess about the important problem of individual ability and efficiency in maturity and to rate ourselves and others chiefly in terms of chronological age. The importance of the study of later maturity whether in connection with industrial conservation or institutional administration is very obvious. To many, especially eager younger persons, Osler's joke is not a joke. But still we have very little actual knowledge of the facts.*

In the study briefly reported in this paper the later age groups were not only included but an effort was made to investigate certain of their abilities with special completeness and in direct comparison with younger groups studied under the same external conditions. The actual research supported by a grant from the Carnegie Corporation was conducted at Stanford University in 1930. Psychological measurements were made on 863 persons; 335 males and 528 females, in age range from 6 to 95 years with as nearly equal numbers in the adult decades as could be secured. About half of these people were 50 years of age or older. Specially outfitted and attractive laboratory rooms, easy of access to those to be examined, were established in two small California cities. Subjects were secured for experiment and measurement through the coöperation and solicitation of lodges, church clubs, etc. For the most part the individuals studied belonged to the upper social strata. No one received personal remuneration. An individual who came to the special laboratory for the two-hour series of tests was credited to the organization that sent him and that organization received payment at a rate proportionate to the individual's age. It was thus possible in part to overcome the difficulty of securing older people, and every person who came had in addition to curiosity an altruistic and generous motive. Coöperation was excellent. The total two-hour session was broken into four half-hour periods with

suitable rest intervals interspersed. The basic abilities: manual motility, reaction speed, visual perception span, judgment of size, judgment of position, code-message substitution, mental and motor learning, immediate memory, ability to overcome memory blocking, and rapid reasoning capacity were tested and measured. Several investigators took part in the research.

The present report is concerned with results from one of the thirty-minute divisions. The two functions studied in this part of the schedule are manual motility and reaction speed. Results for the male subjects only are included in this paper. Apparatus and techniques were so arranged for the measurement of coördination and dexterity that the situations and requirements might seem as natural as possible. The immediate effects of practice were reduced to a minimum.

*Manual Motility.*—1. The examination of motility, in manual reach and grasp, was made for each hand separately. The procedure was as follows: Starting from the clock key (the electric clock was in front and in full view of the subject) the subject reached 6 inches to one side, grasped a round pencil 3 inches long which was standing in a vertical hole $1^1/_2$ inches deep and thrust the pencil into another such opening $1^1/_2$ inches farther away, then returned his hand to the key, thus stopping the clock. In this performance test the subject took his own speed measurement. The clock started when he let go the key and began to reach; it was stopped when he completed the act and returned to the key. Table 1 gives summary results arranged by decades for this apparently simple, but in fact complex, and basic coördination. The total population examined included 331 males; the numbers per age decade are shown. The average score in seconds is the average for the two hands. The average deviation in seconds and the coefficient of variability in per cent are also included. From the table it is apparent that the test in question can be completed

TABLE 1

COÖRDINATION IN MANUAL REACH AND GRASP. (DECADE SCORES FOR BOTH HANDS COMBINED; POPULATION, 331, MALES)

| DECADES | NO. CASES | AVERAGE SCORE, SEC. | AVERAGE DEV. SEC. | C. V. % |
|---|---|---|---|---|
| 80s | 13 | 1.86 | 0.37 | 19.9 |
| 70s | 39 | 1.53 | 0.16 | 10.2 |
| 60s | 52 | 1.39 | 0.16 | 11.2 |
| 50s | 58 | 1.30 | 0.15 | 11.5 |
| 40s | 39 | 1.24 | 0.10 | 8.5 |
| 30s | 40 | 1.23 | 0.12 | 9.4 |
| 20s | 42 | 1.18 | 0.11 | 9.3 |
| 15–19 | 21 | 1.19 | 0.06 | 5.1 |
| 10–14 | 17 | 1.37 | 0.10 | 7.2 |
| 6–9 | 10 | 1.53 | 0.16 | 10.3 |

by men in their 20's on the average in 1.18 seconds and practically as fast by boys in the late teens, aged 15–19 years. Younger and older ages show less speed. The deceleration from the 5th to the 8th decade is regular. But, as far as indicated by the sample, individuals in their 80's are only 50 or 60 per cent slower than the 20-year group. Individual differences are somewhat larger in the later decades. The bottom curve in figure 1 gives the same series of results graphically in terms of semi-decades, thus indicating the degree of regularity in the performance curve in its relation to age. The slope of this curve indicates a rather

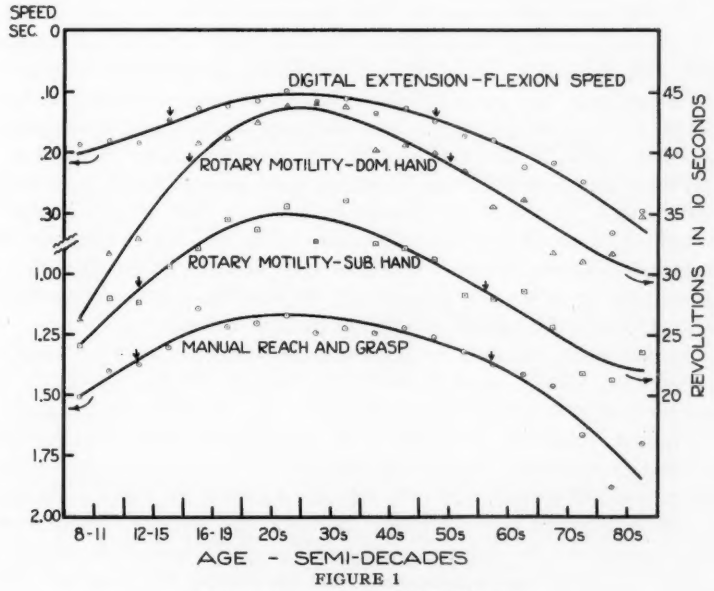


FIGURE 1

Changes in manual motility with age.

rapid improvement in coördination ability from 8 to 18, only a slight change from then until age 50, followed by a decline, quite definite, but I think surprisingly small. Of the 52 men who were 70 years of age or older, the fastest third average at 1.35 seconds, a speed of performance which, as indicated by the short vertical arrows on the curve, is equal to the average ability of 60-year-olds and, at the younger end, of 14-year-olds.**

The results for three other manual motility measures are also charted in figure 1. These are digital extension-flexion speed, and rotary motility for dominant and subordinate hand separately.

2. The extension-flexion measure was made on the forefinger of the

dominant hand. The person lifted his finger from the clock key and brought it down again as quickly as possible. This activity was most efficiently executed by the 25–29-year group who required just less than a tenth of a second. Men 65 to 80 required more than two-tenths and those over 80 three-tenths second. Just as in the period of early development there is a steady increase in ability in this function, so it appears to decline rather steadily from the age of best performance onward. But the averages do not tell the whole story. In the older ages there is great individual variation exhibited by the fact that the best third of the men over 70 have an average which is better than the 50-year-old mean and equals the 14–15-year level.

3. Rotary motility was measured for each hand by means of a small hand drill mechanism fitted with a revolution counter and supported for easy action. The crank handle of this device when revolved described a circle 5 inches in diameter. The load was minimal and the position favorable. Men 25–39 years of age, the best performers of this function, were able to revolve the dominant hand around this 5-inch circle approximately 44 times in ten seconds, but with the subordinate hand they executed about ten revolutions less. The two curves indicate the increasing difference between dominant and subordinate hands with increasing age up to early adulthood and the maintenance of a rather constant difference throughout middle life and later maturity.

*Reaction Speed.*—Results for three reaction speed measures are charted in figure 2. These psychological tasks can be briefly described. 1. The curve at the top represents what has been called pursuit reaction. The subject repeatedly tried to stop the moving index of an electric clock at a certain zero-point position on the dial. The speed was one revolution in one-half second, and each time he watched the index make one or more revolutions before pressing the key. The average displacement error (sometimes the index was stopped too early, sometimes too late) is recorded here as the reaction score, and the smaller this time value the more accuracy is exhibited in this anticipatory reaction response. Ability in this function increases from ages 8 to 18. Thereafter we find but little change until age 30 after which the average displacement error becomes progressively larger, doubling in size by age 80. However, the best third of the older (above 70) group score better than the average 50-year-old man.

2. In the digital reaction test the subject lifted his finger (forefinger of dominant hand) as soon as possible after hearing the buzzing of the electric clock which was started by the experimenter some time (a variable interval) after a warning verbal signal. Lifting the finger stopped the clock. This simple reaction when unpractised requires about two tenths second for men in their 20's. It seems highly significant that there is

retention of this ability into the 6th decade and only slight loss in the 70's and 80's.

3. The foot reaction was of similar nature but here, of course, the key was on the floor and the manipulation was almost identical with that of releasing the gas pedal of an automobile. When the clock was started the subject responded to this auditory signal by lifting the forepart of the foot, using the heel as fulcrum. These results show slightly longer reaction times than are found in responding by lifting the finger and also show a more regular change with increasing age. The results in the semi-decades beyond 70 were too irregular properly to lend themselves to curve fitting.

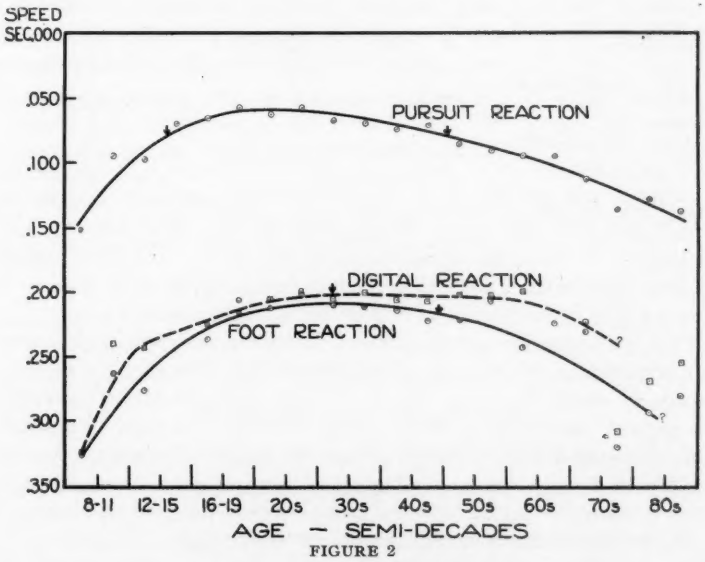


FIGURE 2

Changes in reaction time with age.

Some of the correlations of our motility and reaction scores with age are exhibited in table 2. The functions are named in the left-hand column as in the charts. The number of cases used for each correlation is included. In the second column the raw eta coefficients have been entered. These are for both men and women considered as one group and express the relationship of our curvilinear score diagrams with age throughout the entire age span. In this table these values range from 0.46 to 0.67. The other columns present the product movements ($r$) coefficients for score with age for males considered in four different age groupings: 20–95, 30–95, 40–95, and 50–95 years. All of these correlation

TABLE 2

CORRELATIONS OF MOTOR ABILITY SCORES WITH AGE

| FUNCTION | ETA BOTH SEXES | PRODUCT-MOMENTS ($r$) COEFFICIENT-MALES 20–95 | 30–95 | 40–95 | 50–95 |
|---|---|---|---|---|---|
| No. cases | 866 | 288 | 246 | 206 | 167 |
| 1. Digital extension-flexion speed | 0.604 | 0.599 ±0.03 | 0.574 ±0.03 | 0.514 ±0.03 | 0.457 ±0.04 |
| No. cases | 858 | 283 | 241 | 201 | 162 |
| 2. Manual reach and grasp (both hands) | 0.625 | 0.535 ±0.03 | 0.527 ±0.03 | 0.541 ±0.03 | 0.526 ±0.04 |
| No. cases | 866 | 285 | 243 | 203 | 164 |
| 3. Manual rotary motility, dom. hand | 0.462 | 0.499 ±0.03 | 0.498 ±0.03 | 0.422 ±0.04 | 0.386 ±0.05 |
| No. cases | 868 | 288 | 246 | 206 | 167 |
| 4. Pursuit reaction (anticipatory) | 0.534 | 0.565 ±0.03 | 0.519 ±0.03 | 0.467 ±0.04 | 0.365 ±0.05 |
| No. cases | 469 | 195 | 153 | 115 | 80 |
| 5. Digital reaction (non-anticipatory) | 0.518 | 0.283 ±0.04 | 0.311 ±0.04 | 0.345 ±0.06 | 0.454 ±0.06 |
| No. cases | 469 | 195 | 153 | 115 | 80 |
| 6. Foot reaction (non-anticipatory) | 0.464 | 0.454 ±0.04 | 0.425 ±0.05 | 0.331 ±0.06 | 0.402 ±0.06 |
| No. cases | 439 | 187 | 148 | 110 | 76 |
| 7. Code: letter substitution | 0.675 | 0.537 ±0.04 | 0.623 ±0.04 | 0.654 ±0.04 | 0.407 ±0.06 |

coefficients are statistically significant. All of them agree in indicating score decrement with age in the functions tested. But none of them is so large as to show age alone dictating the score. The score variance in these basic human abilities appears to depend as much or more on the sum total of other factors as it does on chronological age.

That particular age grouping which shows the definitely highest correlation coefficient ($r$) for a given measure indicates thereby the beginning decade in which decrement starts. Digital extension-flexion speed gives the highest correlation in the age range 20 to 95 indicating that there is some decrease in speed between 20 and 30, similarly for pursuit reaction, while with digital reaction the highest correlation is for the age grouping 50 to 95 years. Thus the different measures show different turning points.

Many representative functions need to be surveyed in respect to their changes during man's life span before we can hope to achieve a thorough understanding of ourselves. But scientific theories are always in order. Are the decrements found in the basic motor functions here presented due to neuromuscular deterioration, to less adequate visual and other sensory control, to disuse, to lack of motivation, or to something decidedly different from these? Our experimental conditions were set to provide motivation (I think we were successful); the tasks were selected from among, or closely related to, the *well*-used, and the demands on vision and hearing were slight. You will have noted that when the measure included only the movement, decrement was larger than when the measurement included also the strictly cortical elements of comprehension, selection and readiness for response plus a brief movement. This general dis-

tinction may be made between the motility measures and the reaction tests. It is well known that cortical function is economical of energy and that motor function is spendthrift. A possible theory then for the slower and more difficult action in the old is that neural conservation mechanisms are built up or become more potent, with increasing lifetime. A particular decrement according to this theory would not be chiefly chargeable to a defect in the mechanism but to a positive check on it—a neural governor device protective of the mechanism. The weight of years may be in large part neural inhibition-interference to action. This is perhaps the core, or the basic behavior element, in the caution and proverbial good judgment of the old. Surely the self-depreciation and inferiority attitudes exhibited by the majority of older people in reference to action are scarcely warranted from our data. Decrement appears more in feeling than it exists in fact.

* In 1928 this lack of scientific information was drawn to the attention of various research groups in the United States by Professor Lewis M. Terman and the writer. We prepared a statement, circulated it privately and sought support for investigations, The Carnegie Corporation of New York made a grant to Stanford University to aid such studies. The present paper reports a part of the work thus made possible.

** In figures 1 and 2 the short arrows indicate in each instance the average value for the best one-third of the group 70 years old or older.

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## *THE USE OF ROCK CRYSTAL BY PALEOLITHIC MAN*

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Practically all we know about Paleolithic man's culture is based on his use of materials which do not perish easily with the passage of time. This eliminates to a large degree organic materials which he must have used such as wood, bone, horn and ivory. There remain the inorganic substances with which he gradually became familiar such as the various kinds of stone. For this reason we are accustomed to speak of the oldest known period of man's culture as the Stone Age.

One of the first steps in the ascent of the ladder of culture was taken when man began to find artificial ways of adding to the effectiveness of the powers with which Nature endowed him. He found by experience that a club supplemented arm strength and that the sharp edge or point of a flint chip was more serviceable for cutting and puncturing than nails or teeth. I have been able to list some fory-seven varieties of stone which were em-

ployed during the Stone Age; not all of these were made use of during the Paleolithic Period.

The lithic materials employed by man in prehistoric times belong to three groups: oxides, silicates, and carbonates. Of these three the oxides were much more widely used and were in fact the first to attract man's attention. Of the oxides the species known as quartz was found by experimentation to be especially adapted to the needs of primitive man. The great bulk of Stone-Age tools and weapons are made of cryptocrystalline quartz in the form of flint, chert, chalcedony, hornstone, etc. Paleolithic implements are limited almost wholly to these. In the light of our present knowledge monocrystalline quartz was not in use prior to the Mousterian Epoch.

The common name for monocrystalline quartz is rock crystal; the crystals belong in the trapezohedral class of the rhombohedral division of the hexagonal system. They are transparent and may be without color; they may also take on various colors. Even when too small to be chipped into tools perfect rock crystals were sometimes picked up by Neandertal man and carried to his camp site. Three were found in the Wincqz exploitation at Soignies (Hainault) Belgium. The chipping of tools from rock crystal began with Neandertal man. He was not an artist in the strict sense; that he had an eye for beauty of form and color is evident from even the limited use he made of rock crystal. His technical skill in chipping was worthy of the material on which he worked.

Complete tools of rock crystal dating back to the Mousterian epoch are exceedingly rare. We appreciate all the more the good fortune which has come to our American School of Prehistoric Research in the finding of seven perfect rock crystal tools not only in a single rock shelter but also at one level in that rock shelter (the lower of two Mousterian levels). This shelter is known as *Les Merveilles* and is at Castel-Merle near Sergeac (Dordogne).

Of the seven tools of rock crystal from Les Merveilles (Fig. 1) No. 1, found in 1924, is transparent and tinged with just enough yellow to give it the appearance of a topaz. It is a combination scraper and point. The ventral face is marked by a bulb of percussion and an uneven resin-like surface of fracture. The outer or dorsal face is everywhere reduced by means of chipping except for a small area at the level of the greatest diameter. Half of a scraper of the same quality of rock crystal was found in 1924 in the cultivated field just below and adjoining our leased site.

During the season of 1926 there was found a scraper of purest rock crystal without a tinge of color (No. 2). The portion opposite the edge is perfectly adapted to fit the curve of the forefinger and the dorsal face along the edge has been retouched more than once. The present edge shows the effect of wear; a short bit of it was carried away by a chip accidentally removed in Mousterian times.

Number 3, found in 1928, is amethystine in color. Two of the adjacent hexagonal faces of the original crystal are retained on the part opposite the edge and form a fitting handhold. The edge is retouched along the dorsal

FIGURE 1

Tools of rock crystal and Spanish topaz dug from the rock shelter of Les Merveilles (Dordogne) by students of the American School of Prehistoric Research, Mousterian Epoch.

face only. During the same season Number 4 was found. It had been struck from an exquisite crystal yellowish in tone, but somewhat paler than the parent core of No. 1. The specimen is completely bounded by a series of delicate retouches confined to the dorsal face only.

Numbers 5, 6 and 7 were all found in 1929. Number 5 resembles Number

4 in shape and size and Number 1 in color. Its dorsal face has been completely retouched except for a small area near the base. Here, as in Number 1, the untouched facet represent a bit of surface of fracture made at some preceding time rather than a bit of one of the hexagonal faces of the original crystal.

In so far as quality of crystal is concerned, Number 6 is the poorest of the lot. It is colorless but not flawless. That the crystal mass from which it was struck had been reduced to a pebble by water action is shown in the unworked part opposite the edge and reserved as the handhold. The last and largest of the series of rock crystal tools (No. 7) was likewise struck from a crystal that had been reduced to a pebble; it was not the same pebble as in the preceding case but a much larger one formed from a crystal of great purity—absolutely clear and flawless. On the unretouched end there is a fine bulb of percussion. The dorsal face is completely retouched except at the base and the retained bit of the pebble surface on the lower half of the side opposite the long retouched margin.

The retention of a portion of the pebble surface on two of these rock crystal tools and of a slightly waterworn surface of fracture on two others (1 and 5) may serve as a clue to the location of the original source of rock crystal supply. The nearest source is the region of eruptive rock lying near to and beyond the headwaters of the Vézère in the direction of the Limousin and Puy-de-Dôme. They were carried from this region to Sergeac either by Neandertal man or by the Vézère river. The fact that at least two of the rock crystal tools were struck from pebbles points to their transport by the river. Neandertal man had only to pick up the pebbles in the river bed at the foot of his Abri des Merveilles. They were carried to the rock shelter before the chipping was done just as was the supply of quartzite pebbles for hammerstones. Chips of rock crystal were found in situ in the lower Mousterian level. We know that tools of flint were shaped on the spot because we were able to fit one tool to its parent core which when found was only some 15 cm. removed from the tool itself.

Seven Mousterian tools of rock crystal have also been found at the station of Chez-Pouré (Corrèze). This completes the list of perfect rock crystal tools dating from the Mousterian epoch. Imperfect and fragmentary examples have been reported from the cave of La Chapelle-aux-Saints (Corrèze) and Gudenus cave in Lower Austria.

Upper Paleolithic deposits, especially Solutrean and Magdalenian, have yielded several examples. Solutrean points of the laurel-leaf type, beautifully chipped on both faces, have been found, one in the cave of Le Placard (Charente), one in the cave of Livèyre near Les Eyzies (Dordogne), one in the rock shelter of Badegoule (Dordogne), one from the rock shelter of La Balutie (Dordogne), and one in the rock shelter of Laugerie-Haute (Dordogne). A rock crystal tool dating from the Magdalenian epoch has

been found in the rock shelter of Laugerie-Basse (Dordogne) and two of indifferent workmanship in the station of Le Perron (Haute-Loire).

It will be seen that the foregoing examples of complete rock crystal tools of Paleolithic age all come from three adjoining departments in south central France: Charente, Dordogne and Corrèze. A single center somewhere near the headwater of the Vézère river might well have been the source of rock crystal supply for the Paleolithic sites in question.

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## *REVERSIBLE COAGULATION IN LIVING TISSUE. IX**

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A treatment for morphinism was proposed in the second paper of this series,[1] using sodium rhodanate to peptize the colloids of the central nervous system that are agglomerated by the action of morphine. By demonstrating clearly that the theory works out in living tissue, a sound basis can be laid down for further work of the utmost importance. This study was made in order to pave the way to a greater appreciation of the use of sodium rhodanate in drug addiction. The theory is essentially the same no matter what the addicting drug; although the treatment of addiction to other drugs might vary somewhat.

As a result of their study of morphine addiction in dogs, Plant and Pierce[2] say: "We believe that the observations reported in this paper show there is a marked similarity between morphine addiction and withdrawal in man and in dogs; that this similarity makes the dog a particularly suitable test object for the study of many phases of the morphine problem; that results obtained in experiments on dogs can be applied to the problems of morphine addiction with greater validity than those on any other laboratory animal." Contrary to this, Professor M. Miyajima has informed us[3] that the monkey is the best laboratory animal on which to study drug addiction. Nevertheless, the comparative ease of procuring and caring for dogs led us to choose that animal for these experiments.

Six dogs were obtained and acclimatized to their new surroundings for a little over one month before starting the experiment. This collection of animals included one male collie, designation C1, about two years old; one female collie, designation C2, about two years old; one female mixed hound, designation H1, about one year old; and three female beagles (litter-mates), designations B1, B2, and B3, about two and one-half months old. At the start of the experiment all of the dogs were lively

and in good health. A pre-addiction study revealed that C1 was a wild, brainless dog, neither malicious nor interested in human beings. He would bark continuously at the sky when chained up outside. C2 was a thoroughbred, very quiet and ladylike, with a tendency to be shy. H1 continuously sought human affection. B1, B2, and B3 were typical puppies, full of life and interested in everything.

Morphine was administered daily by the subcutaneous injection of a distilled water solution of morphine sulphate containing one-half grain of the drug in each cc. of solution. In order not to disturb profoundly the general physiological processes of the dogs, addiction was brought about on a small amount of morphine, as shown in table 1. Likewise, the dosage of morphine was kept small enough at all times so that, while the dogs

TABLE 1

| DOG | DAILY MORPHINE DOSAGE AT START | DAILY MORPHINE DOSAGE* AT WITHDRAWAL | WEIGHT AT START | WEIGHT AT WITHDRAWAL |
|---|---|---|---|---|
| C1 | 1.8 mg/kg. | 6.26 mg/kg. | 17.8 kg. | 17.55 kg. |
| C2 | 2.1 | 7.15 | 15.1 | 13.6 |
| H1 | 1.3 | 9.79 | 12.8 | 7.9 |
| B1 | 2.8 | 11.84 | 5.77 | 6.0 |
| B2 | 3.3 | 16.9 | 4.87 | 4.2 |
| B3 | 3.6 | 14.6 | 4.5 | 4.87 |

* A 150-pound man addicted to 5 grains of morphine per day would be taking a daily dose of the drug which amounts to 4.76 mg/kg.

became drowsy at first, the drug did not exert its narcotic effect. Even so the dosage was increased too rapidly in the case of H1 for she lost weight rapidly and came down with a severe case of distemper which seemed to be largely intestinal. Although the data are given for H1, she played no rôle in this study because of her illness. No new symptoms were noticed when the drug was withdrawn from this dog.

The dogs were given the drug daily for several weeks; then C1 was given sodium rhodanate in capsules for two days, following which the morphine was abruptly withdrawn while the peptizing agent was continued. C2 and B3 received no sodium rhodanate. H1, B1, and B2 were given sodium rhodanate in a 10% aqueous solution by subcutaneous injection for five days prior to abrupt withdrawal, after which the sodium rhodanate was continued. The data are given in table 2.

During the period in which the dogs were addicted many of the common symptoms[4] were noticed, i. e., vomiting, salivation, constipation, and comparative lassitude. The vomiting stopped after about three weeks; the salivation began after about two weeks, and continued until two weeks after withdrawal. Reach[5] considers that salivation before and after the injection is an abstinence symptom. Pavlov,[6] on the other

TABLE 2

| DOG | SEX | NUMBER OF DAYS ON MORPHINE | AMOUNT NaCNS BEFORE WITHDRAWAL | AMOUNT NaCNS AFTER WITHDRAWAL | DAYS NaCNS CONTINUED AFTER WITHDRAWAL | TOTAL AMOUNT NaCNS |
|---|---|---|---|---|---|---|
| C1 | M | 38 | 0.8 gm. | 3.4 gm. | 10 | 4.2 gm. |
| C2 | F | 41 | 0 | 0 | 0 | 0 |
| H1 | F | 35 | 1.0 | 0.5 | 5 | 1.5 |
| B1 | F | 41 | 1.1 | 1.3 | 9 | 2.4 |
| B2 | F | 41 | 1.0 | 1.3 | 9 | 2.3 |
| B3 | F | 41 | 0 | 0 | 0 | 0 |

hand, calls it a conditioned reflex. We are inclined to agree with this latter view because increasing the dosage of morphine failed to stop the salivation; also, the amount of salivation decreased markedly when the drug was withdrawn in the case of the control animals. Further, the dogs all began to salivate upon the appearance of the experimenters no matter what the time of day, or whether before or after the morphine injections were made, although visitors did not induce this reaction. At times the reflex vomiting described by Pavlov was observed. For instance, one time while injecting a dog in the sight of another dog, the other dog began to vomit.

One dog, C1, began to gnaw at the wood supports in his cage after four weeks on morphine. This is considered to be a withdrawal symptom;[7] still, on comparatively greater increases in morphine dosage the symptom was not alleviated.

The success of this experiment depended upon the appearance of withdrawal symptoms in the dogs that were not treated with sodium rhodanate, and the non-appearance of withdrawal symptoms in the dogs protected by sodium rhodanate. Reserving C1 for special comment, and ruling out H1 because of distemper, leaves C2, B1, B2, and B3; all four of these dogs were addicted in an identical manner except for dosage. That is, the dosage increase per unit of time, disregarding variation in weight, was the same. The dose was increased 0.0032 gram every other day from the 12th to the 41st day. B1 and B2 were given sodium rhodanate for five days previous to the abrupt withdrawal of the morphine; during this period the regular increase of the morphine was continued. While this was going on B1 and B2 became somewhat depressed, accompanied by a slight loss of appetite, which was reflected by a small decrease of weight.

There were no withdrawal symptoms shown by B1 and B2 upon withholding the drug. They became brighter in two days; their appetites were much better; there were many stools that were not soft; there were no evidences of diarrhea, tremors, or dilated pupils; they showed practically a straight line increase in weight. There was no nervousness. In

other words, no withdrawal symptoms were noticed under the sodium rhodanate treatment.

In contrast to this, C2 and B3 began to show withdrawal symptoms two days after the morphine was abruptly withdrawn. All four of these withdrawals were made on the same day. The second day after the drug was withdrawn C2 showed a desire for it which was manifested by excessive friendliness, in direct opposition to marked shyness during the addiction period. When she learned that she was not to have the drug she became retiring and paid no attention to the experimenters. When the hypodermic syringe was exhibited she nosed it (she is not an inquisitive dog) and became somewhat restless. C2 and B3 had slight diarrhea; the pupils of their eyes were widely dilated, a withdrawal symptom in human beings.[8] C2 had a pronounced head tremor, and a fine body tremor; while B3 had a pronounced body tremor and a slight head tremor. At times B3 would walk around her cage in a nervous aimless way for short intervals. C2 seemed to blow air out with her mouth closed in such a manner as to make her upper lip move rapidly out and in. Both of these dogs lost weight slightly during the first six days of the withdrawal period, in contrast to a gain in weight by those on sodium rhodanate. The withdrawal symptoms subsided twelve days after the morphine was discontinued.

The behavior of these four dogs, C2, B1, B2 and B3, shows very definitely that sodium rhodanate can be used to control the abstinence symptoms resulting from the abrupt withdrawal of morphine from addicted tissues.

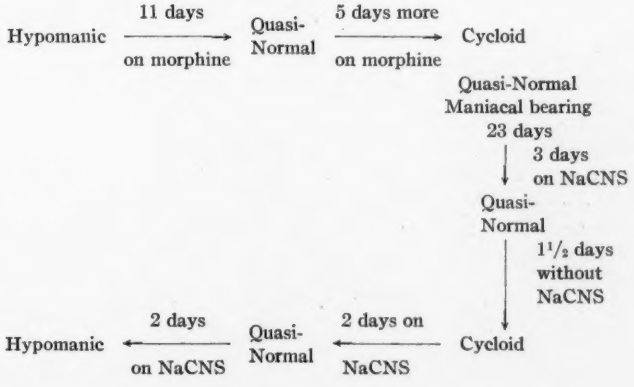
A close study of the actions, reactions and habits of this group of dogs, excepting C1, convinces us that the demand of the habituated tissue for the drug is reflected in the behavior of the animals. One example will illustrate the point aptly. After about three and one-half weeks on morphine, all of the dogs, except C1, began to wag their tails when they were injected. The episode did not begin when they were approached prior to the injection. Likewise, there was neither a response to the preliminary handling nor to the insertion of the needle under the skin. When the liquid began to enter the body from the syringe the dogs began to wag their tails rapidly. There was no such response to the injection of sodium rhodanate during the withdrawal period.

C1 provided a most interesting behavior study; before the start of the experiment he was observed to be highly excitable, scatter-brained, and hard to control. On being approached, he tended to be suspicious, and was without care for human company or affection. He was not a stray dog. He yelped senselessly at the sky for long spells. Eleven days on morphine quieted him down; he became slightly more approachable, less wild and nervous, and ceased barking at the sky. This was a period

when he was comparatively normal. Five more days carried him out of this stage into a different phase wherein he was somewhat restless, very unapproachable, but not mean. He tended to stand faced away from human beings, sometimes with his head in the corner. There was a very obvious hunted, maniacal look in the eyes of this dog during this period. He was so restless and uncontrollable that it was hard to inject him; but upon soothing him the task was sometimes easier. Three days on sodium rhodanate, after withdrawing the morphine, brought C1 back to a normal condition. Then, the sodium rhodanate was stopped for a day and a half, and the dog reverted to the condition which had been relieved by the peptizing agent. Resumption of the sodium rhodanate at first made C1 act more like a normal dog, and then his original scatter-brained condition reappeared.

The following diagram illustrates the reactions that C1 exhibited. The terms hypomanic and cycloid are used to denote agglomeration and more agglomerated, respectively, of certain of the brain colloids, as suggested now by Dr. Lang. Case 6 of their report illustrates a similar effect upon the behavior reactions of man when too much of an agglomerating or peptizing agent is administered.

The only withdrawal symptom shown by C1 was the gnawing which was also noticed during the time that the morphine dosage was being increased the greatest.




Abe[9] explains tolerance, both natural and acquired, addiction, and withdrawal symptoms on the basis of changes in the ratio:

$$\frac{\text{affinity of the tissues for morphine}}{\text{affinity of the blood serum for morphine}}.$$

His conclusions were based on a study of the relative adsorbing power of the tissues and the blood serum for morphine under various conditions. It is a pity that Abe uses narcotics for the treatment of drug addicts rather than taking the shortest route and using peptizing agents; for he has the details of the general theory well worked out.

The general conclusions of this paper are:

1. The theory demands that the rapid peptization of the colloids agglomerated by morphine in living tissue should forestall, or diminish greatly the abstinence symptoms when the drug is withdrawn rapidly.

2. Sodium rhodanate prevented the onset of withdrawal symptoms when morphine was withdrawn abruptly from dogs addicted to morphine.

3. Addicted dogs that were not given sodium rhodanate exhibited withdrawal symptoms when the morphine was abruptly discontinued.

4. Sodium rhodanate counteracts morphine in living tissue.

5. One dog was changed from a hypomanic state to a cycloid state via a quasi-normal state by morphine; sodium rhodanate forced him back to a quasi-normal condition, and finally to the original hypomanic state.

6. Many of the addicted dogs showed definite signs of a desire for the drug.

7. The way is clear for the use of sodium rhodanate in the treatment of drug addiction in human beings.

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[1] Bancroft and Rutzler, *Proc. Nat. Acad. Sci.*, **17**, 186 (1931).

[2] Plant and Pierce, *J. Pharm. Exp. Therap.*, **33**, 356 (1928).

[3] Private Communication.

[4] Plant and Pierce, *J. Pharm. Exp. Therap.*, **33**, 356 (1928); **31**, 210 (1927); Faust, *Archiv. exp. Path. Pharm.*, **44**, 217 (1900).

[5] Reach, *Z. exp. Path. Therap.*, **16**, 321 (1914).

[6] Pavlov, *Conditioned Reflexes*, 35 (1927).

[7] Plant and Pierce, *J. Pharmacol. Proc.*, **31**, 210 (1927).

[8] Lambert, *Am. J. Psychiatry*, **10**, 504 (1930).

[9] Abe, *Revue d'Hygiène*, **52**, 365 (1930).

# FURTHER PROPERTIES OF ABELIAN INTEGRALS ATTACHED TO ALGEBRAIC VARIETIES

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In a recent note[1] I made use of the idea of the product of two constructs in order to obtain certain properties of Abelian integrals attached to algebraic varieties. The present note uses the same method to obtain further properties of such integrals.

## §1.

1. Let $A$, $B$ be two homeomorphic absolute manifolds of $n$ dimensions, and consider the product $A \times B$. On this there is a cycle $\Gamma$ of $n$ dimensions, homeomorphic to $A$ or $B$, corresponding to the transformation between $A$ and $B$ implied by their homeomorphism. If $a_p^i$ $(i = 1, \ldots, R_p)$, where $R_p$ is the $p$th Betti number of $A$, is a base for the $p$-cycles of $A$, and $b_p^i$ $(i = 1, \ldots, R_p)$ is the *corresponding* base for the $p$-cycles of $B$, then

$$a_p^i \times b_{r-p}^j \ (i = 1, \ldots, R_p;\ i = 1, \ldots, R_{r-p};\ p = 0, \ldots r)$$

form a base for the $r$-cycles of $A \times B$. We therefore have

$$\Gamma \approx \sum_{i,j,p} \epsilon_{ij}^p \ a_p^i \times b_{n-p}^j.$$

The coefficients $\epsilon_{ij}^p$ are known,[2] and the matrix $(\epsilon_{ij}^p) = \epsilon^p$ is given by $\epsilon^p = (\overline{a^p})^{-1}$ where $\overline{a_{ji}^p} = a_{ij}^p = a_p^i . a_{n-p}^j$ (the intersection of $a_p^i$ and $a_{n-p}^j$). Now let $\Gamma_r^h$ be the $r$-cycle of $\Gamma$ which is the homeomorph of the cycle $a_r^h$ of $A$.

$$\Gamma_r^h \approx \Gamma . a_r^h \times B$$

$$\approx \sum_{i,j,p} \epsilon_{ij}^p \ a_p^i . a_r^h \times b_{n-p}^j \approx \sum_{i,j,p} (-1)^{pr} \epsilon_{ij}^p a_r^h . a_p^i \times b_{n-p}^j.$$

Let $\quad a_r^h . a_p^i \approx \sum_l \mu_{hil}^{rp} \ a^l{}_{(r+p-n)}$ (or zero if $r + p < n$).

Then $\quad a_r^h . a_p^i . a_{2n-r-p}^j = \sum \mu_{hil}^{rp} \ a^l{}_{(r+p-n)} . a_{2n-r-p}^j.$

The matrices $\quad a_h^{rp} = (a_r^h . a_p^i . a_{2n-r-p}^j),$

and $\quad a^{r+p-n} = (a_{r+p-n}^i . a_{2n-r-p}^j)$

are characteristic matrices of the manifold and are known when the manifold is completely specified. Hence if

$$\mu_h^{rp} = (\mu_{hij}^{rp}),$$

$$\mu_h^{rp} = a_h^{rp}(a^{r+p-n})^{-1}.$$

Now
$$\Gamma_r^h \approx \sum (-1)^{pr} \epsilon_{ij}^{p} \mu_{hil}^{rp} a_{r+p-n}^{l} \times b_{n-p}^{j}$$
$$\approx \sum \lambda_{lj}^{r+p-n} a_{r+p-n}^{l} \times b_{n-p}^{j},$$

where
$$\lambda^{r+p-n} = (-1)^{pr}(\overline{\mu_h^{rp}})\epsilon^{p}$$
$$= (-1)^{pr}(\overline{a^{r+p-n}})^{-1}(\overline{a_h^{rp}})(\overline{a^{p}})^{-1},$$

and hence $\Gamma_r^h$ can be expressed in terms of the base for $r$-cycles of $A \times B$ by a homology in which the coefficients are known.

A more convenient method of writing the result is to put

$$p + q = r.$$

Then
$$\lambda^{p} = (-1)^{(n-q)r}\,\overline{(a^{p})}^{-1}\overline{(a_h^{r,n-q})}(\overline{a^{n-q}})^{-1}.$$

2. Let the manifolds $A$ and $B$ be two Riemann manifolds associated with the algebraic variety $F$ whose equation is

$$F(x_0, x_1, \ldots, x_m) = 0$$

in a (complex) space of $m + 1$ dimensions in which the coördinates are $(x_0, \ldots, x_m)$. $A \times B$ is then a Riemann manifold associated with the ordered product $\Phi$ of the variety by itself, or, if we like, the intersection of the cylinders

$$F(x_0, x_1, \ldots, x_m) = 0$$
$$F(x_0', x_1', \ldots, x_m') = 0$$

in the space of $2m + 2$ dimensions $(x_0, \ldots, x_m, x_0', \ldots, x_m')$.

If
$$\int_{(p)} \sum_{i_q=1}^{m} P_{i_1 i_2 \ldots i_p} dx_{i_1} \ldots dx_{i_p} = \int_{(p)} dP$$

is a $p$-fold integral of a total differential of the first kind attached to $F$, and

$$\int_{(q)} \sum_{j_r=1}^{m} Q_{j_1 \ldots j_q}\, dx_{j_1} \ldots dx_{j_q} = \int_{(q)} dQ$$

is a $q$-fold integral of similar type,

$$\int_{(p+q)} \sum_{i,j} P_{i_1 \ldots i_p} Q'_{j_1 \ldots j_q}\, dx_{i_1} \ldots dx_{i_p}\, dx_{j_1}' \ldots dx_{j_q}', \qquad (1)$$

where the "$Q'$" implies that each $x_i$ is replaced by $x_i'$ in $Q$, is a $(p + q)$-fold integral of a total differential on $\Phi$. The variety of $m$ dimensions

on $\Phi$ which corresponds to the identity transformation on $F$ is given by putting $x_i = x_i'(i = 0, \ldots, m)$ and is represented on $A \times B$ by $\Gamma$. The value of the integral (1) taken over a cycle $\Gamma_r^h$ $(r = p + q)$ of $\Gamma$ is equal to the value of

$$\int_{(p+q)} \sum_{i,j} P_{i_1 \ldots i_p} Q'_{j_1 \ldots j_q} \, dx_{i_1} \ldots dx_{i_p} \, dx_{j_1} \ldots dx_{j_q}$$

taken over the corresponding cycle of $A$, and hence it vanishes if $r > m$. But as (1) is the integral of a total differential of the first kind on $A \times B$ its value is unaltered if $\Gamma_r^h$ is replaced by a homologous cycle, in particular by the cycle

$$\sum_{i,j,p} \lambda_{ij}^p a_p^i \times b_{r-p}^j.$$

Now the value of the integral evaluated over

$$a_p^i \times b_{r-p}^j$$

is $\omega_i^p \times \omega_j^q$, where

$$\omega_i^p = \int_{a_p^i} dP, \text{ and } \omega_j^q = \int_{b_q^j} dQ',$$

and the value over $a_{p'}^i \times b_{r-p'}^j$ $(p \neq p')$ is zero, since this cycle can be deformed into one lying in

$$x_{p'+1} = 0, \ldots, x_m' = 0, x_{r-p'+1}' = 0, \ldots, x_m' = 0,$$

and, as either $p' < p$ or $r - p' < r - p$, the result follows immediately.

Hence we have

$$0 = \sum_{i,j} \lambda_{ij}^p \omega_i^p \omega_j^q.$$

If $\omega^p$ denotes the matrix whose element in the $i^{\text{th}}$ column and the $j^{\text{th}}$ row is the period of the $i^{\text{th}}$ $p$-fold integral of a total differential on $A$ with respect to the cycle $a_p^j$, we have the matrix equation

$$\omega^p \lambda^p \overline{\omega^q} = 0, \tag{2}$$

and there will be one such relation for each of the $R_r$ $r$-cycles of $\Gamma$.

3. The only question which remains is whether the relations (2) are vacuous or not. Consider the simple case in which $m = 2$, so that $F$ is an algebraic surface, and take $p = 1$, $q = 2$. The relation (2) is made up of relations of the form

$$\sum_{i,j} \lambda_{ij}^1 \omega_i^1 \omega_j^2 = 0,$$

where $\omega_i^1$ $(i = 1, \ldots, R_1)$ and $\omega_j^2$ $(j = 1, \ldots, R_2)$ are the periods of a

Picard integral of the first kind and of a double integral of the first kind, respectively, attached to $F$. We know that $\omega_i^2 = 0$ if $a_2^i$ is an algebraic cycle, and what we require to show is that there exist cycles $a_1^i$, $a_2^j$ such that $\omega_i^1 \times \omega_j^2 \neq 0$, and further $\lambda_{ij}^1 \neq 0$. A simple example will show that this is the case, and hence the relation (2) does in fact yield relations between the periods of the $p$- and $q$-fold integrals of total differentials of the first kind on $F$.

Let $C$ and $D$ be two elliptic curves, and denote their 0-cycles; 1-cycles; and 2-cycles, by $c$; $\gamma_1, \gamma_2$; $C$ and $d$; $\delta_1, \delta_2$; $D$, respectively. Then take $F$ as the product of $C \times D$. On $F$ the 0-cycle is $a_0 = c \times d$, the 1-cycles $a_1^1, \ldots, a_1^4$ are $c \times \delta_1$, $c \times \delta_2$, $\gamma_1 \times d$, $\gamma_2 \times d$, the two cycles $a_2^1, \ldots, a_2^6$ are $c \times D$, $\gamma_1 \times \delta_1$, $\gamma_1 \times \delta_2$, $\gamma_2 \times \delta_1$, $\gamma_2 \times \delta_2$, $C \times d$, and the 3-cycles $a_3^1, \ldots, a_3^4$ are $\gamma_1 \times D$, $\gamma_2 \times D$, $C \times \delta_1$, $C \times \delta_2$. The matrices

$$a^1 = \begin{bmatrix} . & . & . & 1 \\ . & . & -1 & . \\ . & 1 & . & . \\ -1 & . & . & . \end{bmatrix}, \qquad a^2 = \begin{bmatrix} . & . & . & . & . & 1 \\ . & . & . & . & -1 & . \\ . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ . & -1 & . & . & . & . \\ 1 & . & . & . & . & . \end{bmatrix},$$

$$(a_2^i \times a_3^j) = \begin{bmatrix} 0 & 0 & c \times \delta_1 & c \times \delta_2 \\ 0 & c \times \delta_1 & 0 & -\gamma_1 \times d \\ 0 & c \times \delta_2 & \gamma_1 \times d & 0 \\ -c \times \delta_1 & 0 & 0 & -\gamma_2 \times d \\ -c \times \delta_2 & 0 & \gamma_2 \times d & 0 \\ \gamma_1 \times d & \gamma_2 \times d & 0 & 0 \end{bmatrix}.$$

Now let $\Gamma_\tau^h$ be the homeomorph of $a_3^1$, then $a_h^{\tau 2}$ is

$$\begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & -1 \\ . & . & 1 & . \\ . & 1 & . & . \end{bmatrix}.$$

and from this it follows that

$$\lambda^1 = \begin{bmatrix} . & . & -1 & . & . & . \\ . & 1 & . & . & . & . \\ 1 & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}.$$

If $(1, \tau)$ are the periods of the integral of the first kind attached to $C$ and

$(1, \sigma)$ those of the integral on $D$, the period matrix for Picard integrals of the first kind on $F$ is

$$\begin{matrix} 1 & \sigma & 0 & 0 \\ 0 & 0 & 1 & \tau \end{matrix}$$

and the periods of the double integral of the first kind are

$$0,\ 1,\ \sigma,\ \tau,\ \tau\sigma,\ 0;$$

and hence if we take the relation arising from the first Picard integral

$$\lambda^1_{13} = -1,\ \omega^1_1 = 1,\ \omega^2_3 = \sigma,$$

and this establishes the required result.

4. If there is a singular transformation of $F$ into itself we shall get a new algebraic cycle $\Gamma$ on $A \times B$ corresponding to this, and we shall have new relations of type (2) arising from each $\Gamma^h_r$ of this $\Gamma$. Lefschetz[3] has shown how to calculate the coefficients $\epsilon'^p_{ij}$ for $\Gamma$ in this case. Making the necessary alterations in our calculations, if

$$\Gamma^h_r \approx \sum_{i,j,p} \mu^p_{ij}\, a^i_p \times b^j_{r-p},$$

$$\mu^p = \lambda^p f^q \quad (r = p + q)$$

when $f^q$ is the matrix of transformation of the $q$-cycles. The new relation is then

$$\omega^p \lambda^p f^q \overline{\omega^q} = 0.$$

But the Hurwitz[4] theory of correspondence tells us that there is a matric $\nu^q$ such that

$$\nu^q \omega^q = \omega^q \overline{f^q},$$

and hence the new relations can be obtained by combining the relations (2) with the Hurwitz relations.

§2.

5. The second application of our method deals with integrals of the second kind attached to an algebraic variety. It is known that if

$$\int \phi_i(x, y)dx \quad (i = 1, \ldots, 2p)$$

are $2p$ independent integrals of the second kind attached to a curve $C$ of genus $p$

$$f(x, y) = 0,$$

and $(x_1, y_1)$ and $(x_2, y_2)$ are two points of $C$, then there exists a certain matrix $(\epsilon_{ij})$, whose elements do not depend on $(x_1, y_1)$ or $(x_2, y_2)$ such that

$$\sum_{ij} \epsilon_{ij}\, \phi_i(x_1, y_1)\, \phi_i(x_2, y_2) = \frac{\partial}{\partial x_1} R(x_1, x_2, y_1, y_2) + \frac{\partial}{\partial x_2} S(x_1, x_2, y_1, y_2)$$

where $R$ and $S$ denote rational functions; and the differentiations are made on the assumption that $(x_1, y_1)$ and $(x_2, y_2)$ lie on $C$. The importance of this result lies in the fact that it gives an algebraic form to the well-known theory of the "interchange of argument and parameter" for normal integrals of the third kind.[5]

The existence of such a result is obvious from certain geometrical considerations, but not its form. The argument is as follows. We take $C$ as the variety $F$ of §1 and consider the surface $\Phi$. Denote the integrals of the second kind attached to $A$ by $U_1, U_2, \ldots, U_{2p}$, and let the corresponding ones attached to $B$ be $V_1, V_2, \ldots, V_{2p}$. Then clearly $U_i \times V_j$ is an integral of the second kind attached to $\Phi$. Now for $\Phi$

$$R_2 = 4p^2 + 2.$$

Consider the Picard number $\rho$. This is in general equal to 3, for there are three independent curves on it $x = \text{const.}, x' = \text{const.}\ x = x'$. Therefore there are

$$R_2 - \rho = 4p^2 - 1$$

independent integrals of the second kind on $\Phi$, and hence the $4p^2$ integrals $U_i \times V_j$ are not all independent. There must therefore be a matrix $(\epsilon_{ij})$ such that

$$\sum_{ij} \epsilon_{ij}\, U_i \times V_j$$

is improper.

6. This suggests generalizations of the result. In the first place, if there are $k$ singular correspondences on the curve $C$, to each of these there corresponds a curve on $\Phi$ (or rather two curves on $\Phi$, which are interchanged when we interchange $x$ and $x'$). Hence $\rho$ is increased, so that to each correspondence there corresponds a new bilinear expression in $U$, $V$ which is improper. Again, the arguments can be extended to any variety and give similar results for varieties of $m$ dimensions. The application of the results to function theory is, however, not obvious.

By means of our topological methods we are able to give exact forms to the bilinear expressions which are improper, and this paragraph is devoted to finding these forms. We proceed with the most general case and use the notation of §1.

7. Consider on $A \times B$ the algebraic cycle of $2m$ dimensions $\Gamma$ which represents any correspondence between $A$ and $B$,

$$\Gamma \approx \sum_{i,j,p} \epsilon_{ij}^{p}\, a_p^i \times b_{2m-p}^j.$$

From the formula of Lefschetz, already quoted,

$$\epsilon^p = (\bar{a}^p)^{-1} f^{2m-p}$$

where $f^{2m-p}$ is the matrix of transformation of the $(2m-p)$-cycles. Let $\Gamma_1$ stand for the cycle

$$\Gamma_1 \approx \sum_{ij} \epsilon_{ij}^m a_m^i \times b^j m.$$

Then $(\Gamma_1.\Gamma_1) = \gamma = (-1)^m \sum_{i,j,h,k} \epsilon_{ij}\epsilon_{hk} a_{ih} b_{jk}$ (where the affix $m$ is omitted, as this is the only one which is required in the sequel).

Writing $$\Gamma_{rs} = \Gamma_1 \sum_{ij} \epsilon_{ij} a_{ir}\, b_{js} - (-1)^m \gamma a_m^r \times b_m^s,$$

we have

$$(\Gamma_{rs}.\Gamma_1) = 0, \quad \sum \epsilon_{rs}\, \Gamma_{rs} \approx 0.$$

Let $U_1, \ldots, U_{rm}$ be the integrals of the second kind attached to $A$ which have unit periods on $a_m^1, \ldots, a_m^{Rm}$, respectively, and all their other periods equal to zero, and let $V_1, \ldots, V_{Rm}$ be the corresponding integrals of $B$. Some of these may be improper, but this does not affect the argument. The period of $U_a \times V_b$ on $\Gamma_{ab}$ is

$$\sum_{ij} \epsilon_{ab}\, \epsilon_{ij}\, a_{ia}\, b_{jb} - (-1)^m \gamma$$

and on any other cycle $\Gamma_{rs}$ it is

$$\sum_{ij} \epsilon_{ab}\, \epsilon_{ij}\, a_{ir}\, b_{js}.$$

Hence the period of $$\sum_{k,a,b} f_{ak}\, a_{kb}\, U_a \times V_b$$

on $\Gamma_{rs}$ is

$$\begin{aligned} &\sum_{a,b,i,j,k} f_{ak}\, a_{kb}\, \epsilon_{ab}\, \epsilon_{ij}\, a_{jr}\, b_{js} - (-1)^m \gamma \sum_k f_{rk}\, a_{ks} \\ &= \sum_{a,b,j,k} f_{ak}\, a_{kb}\, \epsilon_{ab}\, f_{rj}\, b_{js} - (-1)^m \gamma \sum_k f_{rk}\, a_{ks} \\ &= \sum_j f_{rj}\, a_{js} \Big[ \sum_{akb} f_{ak}\, a_{kb}\, \epsilon_{ab} - (-1)^m \gamma \Big], \text{ (since } a = b) \\ &= 0. \end{aligned}$$

8. Now let us take as our base for cycles of $2m$-dimensions in $A \times B$, $R_m^2 - 1$ of the cycles $\Gamma_{rs}$, the cycle $\Gamma_1$, and the cycles $a_i^h \times b_{2m-i}^k$ $(i \neq m)$. These last cycles can all be submerged in the manifold of $A \times B$ which corresponds to a section of $\Phi$, and $\Gamma_1$ is homologous to the difference between an algebraic cycle and these cycles. Further all these cycles meet $\Gamma_{rs}$ in a number of points algebraically equal to zero. Hence[6] if an integral of the second kind has zero periods on all the cycles $\Gamma_{rs}$ it is improper. Therefore

$$\sum_{k,a,b} f_{ak}\, a_{kb}\, U_a \times V_b$$

is improper. We write this

$$f^m a^m VU$$

In general if $u_1, \ldots, u_{Rm}$ is any system of integrals of the second kind on $A$, and $v_1, \ldots, v_{Rm}$ is the corresponding system on $B$

$$U = \omega^{-1}u, \quad V = \omega^{-1}v$$

where $\omega$ is the period matrix of the integrals and we have the result that if $\alpha$ is a matrix

$$\alpha = \bar{\omega}^{-1}f^m a^m \bar{\omega}^{-1}$$

$$\sum_{ij} \alpha_{ij} u_i \times v_j$$

is improper.

9. It is possible to deduce a similar relation between the $p$- and $q$-fold integrals of total differentials of the second kind by considering the cycle $\Gamma_r^k$ of §1. Until, however, some application of such a result arises there is no point in carrying through the analysis, which does not introduce any new idea.

[1] Hodge, *J. Lon. Math. Soc.*, **5**, p. 283 (1930).
[2] Lefschetz, *Colloquium Lectures on Topology*, p. 266 (1930).
[3] Loc. cit.
[4] Hurwitz, *Mat. Ann.*, **28**, 561–585 (1887).
[5] Cf. Baker, *Abel's Theorem and the Allied Theory*, p. 185.
[6] Lefschetz, *Trans. Amer. Math. Soc.*, **22**, 337 (1921).

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## *PROOF OF A RECURRENCE THEOREM FOR STRONGLY TRANSITIVE SYSTEMS*

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Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of $n$ differential equations of the first order, valid in a closed analytic $n$-dimensional manifold without singularity, $M$. The points of $M$ are taken to be represented by a finite number of such sets of variables $(x)$ in overlapping domains. For definiteness, the right-hand members $X_i$ as well as the transformations of connection between the sets $(x)$ are taken to be analytic. Finally it will be assumed that there is a volume integral invariant, $\int dx_1\, dx_2 \ldots dx_n$ in suitable coördinates.

If we select any $(n-1)$ dimensional analytic surface $\sigma$ in $M$, which cuts the trajectories in one and the same sense throughout at an angle $\theta \geqq d > 0$, the points of $\sigma$ whose trajectories cut $M$ infinitely often as the time $t$ increases and decreases, fill all of $\sigma$ save at most a set of measure 0 in the sense of Lebesgue. This is essentially the significance of the classic work of Poincaré on the recurrence of trajectories.[1]

Now it is probably true that in general such systems (1) are *strongly transitive* in the sense that any measurable set of complete trajectories in $M$ has either the measure 0 or $V$, where $V$ is the total volume of $M$. The fact that such strong transitivity may be realized has been shown in a simple example by E. Hopf, who has first defined this type of transitivity.

We propose in this note to prove the following simple recurrence theorem, if $t_n$ denotes time of the $n$th crossing of $\sigma$ by a trajectory which issues from a point $P$ of $\sigma$, then we have, for a certain constant $\tau$,

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \tau, \tag{1}$$

for all points $P$ save those which belong to a set of measure 0. In other words there is a fixed "mean time" of crossing on a general trajectory.

Very recently von Neumann,[2] by an application of abstract integral equation theory in a direction suggested by Koopman,[3] has obtained results which would show that $t_n(P)/n$ converges *in the mean* toward $\tau$; but this does not show convergence nor a mean time in the usual sense. E. Hopf[2] has established his results directly.

I propose to establish (1) here, and in the following note to establish a general recurrence theorem and thence the "ergodic theorem."

The method of proof is one which I tried to use nearly ten years ago in order to show that there was some uniformity of recurrence when there was merely regional transitivity. That attempt would seem to have failed because the hypothesis was not exacting enough. It is to be remarked that the demonstration of the strong transitivity condition in any except very simple cases appears to be extraordinarily difficult.

Consider an "infinitesimal" cylinder made up of arcs of trajectories with a base $d\sigma$ at $P$ in $\sigma$, and of height $dn$ normal to $d\sigma$. Its volume is then $d\sigma dn$ or $v\cos\vartheta d\sigma dt$, where $v$ denotes the velocity, and $dt$ denotes the corresponding time.

Suppose now that the tube of trajectories with this base $d\sigma$ ($t$ increasing) cuts $\sigma$ again for a first time in the base $d\bar{\sigma}$. A doubly closed tube is thus formed having a total volume $t(P)v\cos\vartheta d\sigma$, where $t(P)$ stands for the time interval between the crossing at $P$ and at $\overline{P}$. Let $t$ increase further by $dt$; the tube then advances to a new position, differing from the former in that the cylinder of volume $v\cos\vartheta d\sigma dt$ has been subtracted, and the cylinder of volume $\bar{v}\cos\bar{\vartheta} d\bar{\sigma} dt$ has been added. But these volumes are

equal, since volumes are conserved. In consequence if we designate the analytic point function $v \cos \theta > 0$ by $\omega(P)$ it is clear that $\omega(P)d\sigma = \omega(\overline{P})d\overline{\sigma}$; in other words, $\int \omega(P)d\sigma$ is conserved by the $(n-1)$-dimensional transformation $T(P)$ which takes $P$ in $\sigma$ to $\overline{P}$ in $\sigma$.

According to the result of Poincaré, the transformation from $P$ to $\overline{P}$ is one-to-one in $\sigma$ except at a set of points of measure 0. Of course, $t(P)$ is defined except at such points. More precisely, $\sigma$ may be broken up into a numerable set of open continua, in which the transformation $T(P)$ and the function $t(P)$ are analytic, together with a further set of measure 0.

If the time to the $n$th crossing of $\sigma$ be denoted by $t_n(P)$ (defined except for a set of measure 0), we have the fundamental functional identity

$$t_n(P) = t(T^{n-1}(P)) + t_{n-1}(P), \tag{2}$$

which states that the time to the $n$th crossing is the time beyond the $(n-1)$th crossing together with the time to the $(n-1)$th crossing. Here $T^k(P)$ denotes the $k$th transformed point of $P$. By successive use of the above identity we derive further

$$t_n(P) = t(T^{n-1}(P)) + t(T^{n-2}(P)) + \ldots + t(P). \tag{3}$$

From this equation we obtain

$$\int_\sigma t_n(P)dP = \int_\sigma t(T^{n-1}(P)dP + \ldots + \int_\sigma t(P)dP, \tag{4}$$

where $dP$ stands for the $(n-1)$-dimensional volume element $\omega(P)d\sigma$.

But $\int t(P)dP$ extended over $\sigma$ is the total volume $V$ of $M$, according to the hypothesis of strong transitivity. For, this integral represents the measure of all the trajectories which issue from $\sigma$, and the remaining measurable set of trajectories is therefore of measure 0 by this hypothesis. Moreover, since $\int dP$ is conserved by $T$, and $T$ transforms $\sigma$ into itself except over a set of measure 0, we have

$$\int_\sigma t(T^k(P))dP = \int_\sigma t/T^{k-1}(P)dP = \ldots = V.$$

Thus (4) gives us

$$\frac{\int_\sigma t_n(P)dP}{n\int_\sigma dP} = \frac{V}{\int_\sigma v\cos\vartheta d\sigma} = \alpha. \tag{5}$$

In other words the *mean time* of the $n$th crossing of $\sigma$ is precisely the ratio $\alpha$ of the total volume to the rate of flux across the surface $\sigma$.

Now consider the set $S_\delta$ of points $P$ such that for a definite $\delta > 0$ and for infinitely many values of $n$, we have

$$t_n(P) > n(\alpha + \delta) \quad (n = 1, 2, \ldots, n) \tag{6}$$

The set $S_{k,\delta}$ of points $P$ for which this inequality holds for *some* $n \geqq k$ is a measurable set, which diminishes (or at least does not increase) with

increase of $k$, toward the limiting measurable set $S_\delta$. Moreover, the set $S_\delta$ has the property of invariance under $S$: $T(S) = S$. Hence $S_\delta$ has either the measure 0 or that of $\sigma$, for, according to the hypothesis of strong transitivity, the measure of the trajectories through $S_\delta$ is $\int_{S_\delta} t(P)dP = 0$ or $V$.

Similarly the set $S'_\delta$ of points $P$ such that for a definite $\delta > 0$ and for infinitely many values of $n$, we have

$$t_n(P) < n(\alpha - \delta) \quad (n = 1, 2, \ldots), \tag{7}$$

is an invariant measurable set of measure 0 or $\sigma$.

If $S_\delta$ and $S'_\delta$ are both of measure 0 for $\delta$ arbitrarily small, we would conclude at once that, for almost all points $P$ of $\sigma$, and for $n$ sufficiently large,

$$n(\alpha - \delta) < t_n(P) < n(\alpha + \delta),$$

no matter how small $\delta$ be taken. In this case, of course, the stated theorem is true.

If this is not the case, suppose for example that $S_\delta$ has the measure of $\sigma$ for some $\delta > 0$. Certainly in that event, the set $S_{k,\delta}$ for $k = 1$ will also have the measure of $\sigma$, since $S_{1,\delta}$ is the set for which (6) holds for some $n$.

Now $S_{1,\delta}$ can be broken up into the sequence of distinct classes $U_1$, $U_2$, ... of points $P$ defined as follows:

$U_1$ : $t(P) > \alpha + \delta$;
$U_2$ : $t_2(P) > 2(\alpha + \delta)$, $P$ not in $U_1$;
$U_3$ : $t_3(P) > 3(\alpha + \delta)$, $P$ not in $U_1$ or $U_2$;
.....

In consequence, if $P$ is a point of $U_k$ we have

$$t_k(P) > k(\alpha + \delta) \quad t_l(P) \leqq l(\alpha + \delta) \quad (1 \leqq l < k)$$

whence, by subtraction and use of (3),

$$t_{k-l}(T^l(P)) > (k - l)(\alpha + \delta).$$

We infer that, if $P$ is a point of $U_k$, then $T^l(P)$ for $l < k$ is a point of one of the sets $U_{k-l}$, $U_{k-l-1}$, ..., $U_1$. It follows that, as $l$ increases, $T^l(P)$ falls successively in sets $U_{k-1}$, ..., $U_1$, with lower subscripts, not more than $k - 1$ points being required before a point of this set falls in $U_1$.

Thus it is seen that one may separate $U_k$ into $k - l$ distinct measurable sets $U_{kj}$ ($j = 1, 2, \ldots, k - 1$), such that, if $P$ lies in $U_{kj}$, the points $T(P)$, ..., $T^j(P)$ fall in $U_i$'s with decreasing subscripts $i < k$, the last point $T^j(P)$ only being in $U_1$.

The measurable sets in $U_k$, ..., $U_1$,

$$U_{kj}, T(U_{kj}), \ldots, T^{k-1}(U_{kj}) \quad (j = 1, 2, \ldots . k - 1), \tag{8}$$

are all distinct from one another. In fact, if there were a point $P$ in common to

$$T^{j_1'}(U_{kj_1}) \text{ and } T^{j_2'}(U_{kj_2}) \qquad (j_1' \leqq j_2'),$$

the transformation $T^{-j_1'}$ would give a corresponding point in common to

$$U_{kj_1} \text{ and } T^{j_2'-j_1'}(U_{kj_2}).$$

But this is obviously not possible for $j_2' - j_1' > 0$; and is only possible for $j_2' - j_1' = 0$ if $j_1 = j_2$. Hence there are no such common points.

Next let us consider the measurable part of $U_{k-1}$ made up of points $P$ *not* in any such set $U_{kj}$. This part may be likewise separated in measurable parts $U_{k-1,j}$ $(j = 1, \ldots, k-2)$ such that if $P$ lies in $U_{k-1,j}$, then $T(P), \ldots, T^j(P)$ fall in sets $U_i (i < k-1)$ with decreasing subscripts, the last point $T^j(P)$ only being in $U_1$.

The sets

$$U_{k-1,j},\ T(U_{k-1,j}),\ \ldots,\ T^{k-2}(U_{k-1,j}) \qquad (j = 1, \ldots, k-2) \qquad (9)$$

so obtained are again distinct from one another. Furthermore they are entirely distinct from the previous sets. For if there were a point $P$ in common to

$$T^{j_1'}(U_{k,j_1}) \text{ and } T^{j_2'}(U_{k-1,j_2}),$$

a transformation by $T^{-j}$, where $j$ is the lesser of the integers $j_1'$, $j_2'$ or else is their common value, would give a corresponding point common to

$$U_{k,j_1} \text{ and } U_{k-1,j_2} \text{ if } j = j_1' = j_2',$$

or common to

$$U_{kj_1} \text{ and } T^{j_2'-j_1'}(U_{k-1,j_2}) \text{ if } j = j_1' < j_2',$$

or common to

$$T^{j_1'-j_2'}(U_{kj_1}) \text{ and } U_{k-1,j_2} \text{ if } j = j_2' < j_1'.$$

The first and second cases are obviously impossible. The last case could only arise for $j_1' - j_2' = 1$; but this is also impossible since $U_{k-1,j_2}$ $(j_2 = 1, \ldots, k-1)$ falls in the part of $U_{k-1}$ not in $T(U_k)$ and so not in any $T(U_{kj_1})$.

Proceeding in the same manner, we may define a set of entirely distinct measurable sets

$$\begin{array}{ll} U_{k,k-1},\ U_{k,k-2} \cdots & U_{k,1}, \\ \cdots\cdots U_{k-1,k-2}, \cdots & U_{k-1,1}, \\ & \cdots \\ & U_{21}, \\ & U_{10}, \end{array}$$

of which the last $U_{10}$ consists of the points of $U_1$ not in any preceding set. These will have the property that the finite set of measurable sets

$$T^m(U_{l,j}) \quad (l = 2, \ldots k, j = 1, 2, \ldots l-1, m = 0, 1, \ldots l-1)$$

are entirely distinct from one another and, together with $U_{10}$, exhaust $S_{1,\delta,k} = U_1 + U_2 + \ldots + U_k$.

Consider now the integral

$$\int_{S_{1,\delta,k}} t(P)dP = \sum_{j,l,m} \int_{T^m(U_{l,j})} t(P)dP + \int_{U_{10}} t(P)dP.$$

This integral may be written

$$\sum_{j,l} \int_{U_{lj}+\ldots+T^{l-1}(U_{lj})} t(P)dP$$

which is the same as

$$\sum_{j,l} \int_{U_{l,j}} t_l(P)\, dP.$$

Since $U_{l,j}$ is a part of $U_l$, each partial integral in $\sum$ exceeds $l(\alpha + \delta) \int_{Ulj} dP$, by definition of $U_l$. This quantity is the same as

$$(\alpha + \delta) \int_{U_{l,j}+\ldots+T^{l-1}(U_{l,j})} dP$$

since $\int dP$ is conserved by $T$. Likewise, $\int_{U_{10}} t(P)dP$ exceeds $(\alpha + \delta) \int_{U_{10}} dP$. Hence we deduce the inequality

$$\int_{S_{1,\delta,k}} t(P)dP > (\alpha + \delta) \int_{S_{1,\delta,k}} dP$$

for $k = 1, 2, \ldots$. But, inasmuch as $S_{1,\delta}$ has the measure of $\sigma$ and is the limit of $S_{1,\delta,k}$ for $k = 1, 2, \ldots$, we would then conclude

$$\int_\sigma t(P)dP \geqq (\alpha + \delta) \int_\sigma dP,$$

which is manifestly impossible.

Evidently the argumentation just given applies equally well for any numerable set of distinct measurable elements of surface, $\sigma$, which make an angle $\theta > d > 0$ with the trajectories and have a finite $\int dP$.

Thus the theorem is proved not only for any single surface $\sigma$ but for any such measurable aggregate.

[1] *Méthodes nouvelles de la Mécanique Celeste*, t. 3.

[2] Not yet published.

[3] "Hamiltonian Systems and Transformations in Hilbert Space," these PROCEEDINGS, 315–318 (May, 1931).

# PROOF OF THE ERGODIC THEOREM

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Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots x_n) \qquad (i = 1, \ldots n)$$

be a system of $n$ differential equations valid on a closed analytic manifold $M$, possessing an invariant volume integral, and otherwise subject to the same restrictions as in the preceding note, except that the hypothesis of strong transitivity is no longer made.

We propose to establish first that, without this hypothesis, we have

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \tau(P) \qquad (1)$$

for all points $P$ of the surface $\sigma$ save for points of a set of measure 0. In other words, there is a "mean time $\tau(P)$, of crossing" of $\sigma$ for the general trajectory.

The proof of the "ergodic theorem," that there is a time-probability $p$ that a point $P$ of a general trajectory lies in a given volume $v$ of $M$, parallels that of the above recurrence theorem, as will be seen.

The important recent work of von Neumann (not yet published) shows only that there is convergence *in the mean*, so that (1) is not proved by him to hold for any point $P$, and the time-probability is not established in the usual sense for any trajectory. A *direct* proof of von Neumann's results (not yet published) has been obtained by E. Hopf.

Our treatment will be based upon the following *lemma:* If $S_\lambda[S'_\lambda]$ is a measurable set on $\sigma$, which is invariant under $T$, except possibly for a set of measure 0, and if for any point $P$ of this set

$$\lim_{n=\infty} \sup \frac{t_n(P)}{n} \geqq \lambda > 0 \left[ \lim_{n=\infty} \inf \frac{t_n(P)}{n} \leqq \lambda > 0 \right] \qquad (2)$$

then

$$\int_{S_\lambda} t_n(P)dP \geqq \lambda \int_{S_\lambda} dP \quad \left[\int_{S'_\lambda} t(P)dP \leqq \lambda \int_{S_\lambda} dP. \right. \qquad (3)$$

We consider only the first case, for the proof of the second case is entirely similar. In analogy with the preceding note, define the distinct measurable sets $U_1, U_2, \ldots$ on $S_\lambda$ so that for $P$ in $U_n$

$$t_n(P) > n(\lambda - \epsilon) \quad (P \text{ not in } U_1, U_2, \ldots, U_{n-1}) \qquad (4)$$

The quantity $\epsilon > 0$ is taken arbitrarily. It is, of course, clear that for every point $P$ of $S_\lambda$

$$t_n(P) > n(\lambda - \epsilon)$$

for infinitely many values of $n$, so that all such points belong to at least one of the sets $U_1, U_2, \ldots$. Now, by the argument of the earlier note, we infer

$$\int_{S_\lambda^k} t(P)dP > (\lambda - \epsilon) \int_{S_\lambda^k} dP$$

where $S_\lambda^k = U_1 + U_2 + \ldots + U_k$. But $S_\lambda^k$ is, for every value of $k$, a measurable part of the invariant set $S_\lambda$ and increases toward a limit $U_1 + U_2 + \ldots$ which contains every point of $S_\lambda$. Consequently we obtain by a limiting process

$$\int_{S_\lambda} t(P)dP \geqq (\lambda - \epsilon) \int_{S_\lambda} dP$$

for any $\epsilon > 0$, whence the inequality of the lemma.

The recurrence theorem stated results directly from this lemma.

Consider the measurable invariant set of points $P$ on $\sigma$ for which

$$t_n(P) \geqq n\lambda \tag{5}$$

for infinitely many values of $n$ (see the preceding note). This is a set $S_\lambda$ to which the lemma applies. Similarly the set of points $P$ on $\sigma$ for which

$$t_n(P) < n\lambda \tag{6}$$

for infinitely many values of $n$ is a set $S_\lambda'$ of the kind specified in the lemma.

The set $S_\lambda$ diminishes and the set $S_\lambda'$ increases with $\sigma$, and both sets taken together exhaust $\sigma$. The measure of the set $S_\lambda$ must tend toward 0 as $\lambda$ increases. Otherwise it would tend toward an invariant measurable set of positive measure, S*, for which the inequality of the lemma holds for $\lambda = \Lambda$, an arbitrarily large positive quantity, and we should infer

$$\int_{S^*} t(P)dP \geqq \Lambda \int_{S^*} dP$$

for any $\Lambda$, which is absurd. Moreover, when $\lambda$ tends toward 0, $S_\lambda$ becomes vacuous, since there is a least time of crossing, $\lambda_0$. In a similar way, $S_\lambda'$ increases with $\lambda$ from a set of zero measure for $\lambda < \lambda_0$ toward the set $\sigma$.

If then $S_\lambda$ and $S_\lambda'$ are not essentially complementary parts of $\sigma$, one decreasing, the other increasing, they must, for certain values of $\lambda$, have a common measurable component $S_\lambda^*$ of positive measure, also invariant under $T$.

Consider the set of points belonging to $S_\lambda^*$ such that

$$t_n(P) > n\mu \qquad (\mu > \lambda)$$

for infinitely many values of $n$. These form an invariant measurable subset $S_{\lambda\mu}^*$ of $S_\lambda^*$, which must be of measure 0 for any such $\mu$. Otherwise the inequalities of the lemma would give us simultaneously

$$\int_{S_{\lambda\mu}^*} t(P)dP \geqq \mu \int_{S_{\lambda\mu}^*} dP, \quad \int_{S_{\lambda\mu}^*} t(P)dP \leqq \lambda \int_{S_{\lambda\mu}^*} dP,$$

which are mutually contradictory.

Hence we infer that all of the points $P$ of $S_\lambda^*$ save for a set of measure 0, satisfy the inequality

$$t_n(P) \leqq n\mu$$

for any $\mu > \lambda$ and for $n = n_P$ sufficiently large, that is,

$$\lim_{n=\infty} \sup \frac{t_n(P)}{n} \leqq \lambda.$$

Likewise we infer that for all of the points of $S_\lambda^*$, save for a set of measure 0, we have

$$\lim_{n=\infty} \inf \frac{t_n(P)}{n} \geqq \lambda.$$

It follows then that for points $P$ of $S_\lambda^*$, with the usual exception,

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \lambda. \tag{7}$$

Two such sets $S_\lambda^*$ belonging to different $\lambda$'s are evidently distinct except for a set of measure 0. Hence there can exist only a numerable set $S_{\lambda_i}^*(i = 1, 2, \ldots)$ of such sets since each has a positive measure. Except for these values $\lambda_i$ of $\lambda$, $S_\lambda'$ and $S_\lambda$ are complementary parts of $\sigma$ aside from a set of measure 0.

Choose now any two values of $\lambda$, say $\lambda,\mu$ with $\lambda < \mu$, not belonging to this numerable set, and consider the points of $S_\lambda$ which do not belong to $S_\mu$. These form an invariant measurable set $S_{\lambda,\mu}$, such that for any point $P$ of this set

$$\lambda \leqq \lim_{n=\infty} \sup \frac{t_n(P)}{n} \mu \tag{8}$$

and also

$$\lambda \leqq \lim_{n=\infty} \inf \frac{t_n(P)}{n} \leqq \mu, \tag{9}$$

since $S_{\lambda,\mu}$ is essentially identical with the part of $S_\mu'$ not in $S_\lambda$. We infer

then that $t_n(P)/n$ oscillates between $\lambda$ and $\mu$ as $n$ tends toward $\infty$, for all points $P$ of $S_{\lambda,\sigma}$ except a set of measure 0.

By choosing a set of values such as $\lambda,\mu$ sufficiently near together we infer then that for all of the points of $\sigma$ except a set of measure 0, the oscillation of $t_n(P)/n$, as $n$ becomes infinite, is less than an arbitrary $\delta > 0$.

Obviously then the stated recurrence theorem is true.

It should also be noted that if $t_n/P$ denotes the time to the $n$th crossing as time decreases, the same result holds if $n$ tends toward $\pm\infty$, *with the same limit* except for a set of points $P$ of measure 0. This follows at once from the fact that (8) may be written

$$\lambda \leqq \lim_{n=-\infty} \sup \frac{t_n(P)}{n} \cdot \leqq \mu,$$

where $P$ of $S_{\lambda,\mu}$ is replaced by $T^n(P)$; and (9) may be given a corresponding form.

This theorem of recurrence admits of certain evident extensions. In the first place there is no need to restrict attention to the analytic case. Moreover, instead of a single surface $\sigma$, any measurable set $\sigma^*$, imbedded in a numerable set of distinct ordinary surface elements with $v\cos\theta > d > 0$, throughout, will serve, in which case $t^*(P)$ denotes the time from $P$ on $\sigma^*$ to the first later crossing of $\sigma^*$.

In order to prove the "ergodic theorem" we observe first that a set $\sigma^*$ can be found which cuts every trajectory except those corresponding to equilibrium and others of total measure 0. This is possible; for a numerable set of distinct ordinary surface elements $\sigma_1, \sigma_2, \ldots$ with $v\cos\vartheta > d > 0$ can be found which cut every trajectory not corresponding to equilibrium. If we define $\sigma_k$ as the limit of

$$\sigma_1 + \sigma_{12} + \sigma_{123} + \ldots + \sigma_{1\ldots k}$$

where $\sigma_{12}$ denotes the set of points $P$ of $\sigma_2$ not on a trajectory cutting $\sigma_1$, $\sigma_{123}$ denotes the set of points of $\sigma_3$ not on a trajectory cutting $\sigma_1$ or $\sigma_2$, etc., it will have the desired properties.

Now let $v$ denote any "measurable" volume in the manifold $M$, and let $\bar{t}(P)$ denote the interval of time during which the point on the trajectory which issues from $P$ on such a set $\sigma^*$ lies in $v$ before the point $T(P)$ of $\sigma^*$ is reached. Thus $\bar{t}(P) \leqq t(P)$ in all cases. In addition, $\bar{t}_n(P)$ satisfies the same functional equation as $t(P)$

$$\bar{t}_n(P) = \bar{t}(T^{n-1}(P)) + \bar{t}_{n-1}(P).$$

Hence the same reasoning as before is applicable to show that, except for a set of points $P$ of measure,

$$\lim_{n=\pm\infty} \frac{\bar{t}_n(P)}{n} = \bar{t}(P),$$

where $\bar{\tau}(P) \leqq \tau(P)$; while at the same time, of course,

$$\lim_{n = \pm \infty} \frac{t_n(P)}{n} = \tau(P) > 0.$$

We conclude that the following "ergodic theorem" holds:

*For any dynamical system of type* (1) *there is a definite "time probability" p that any moving point, excepting those of a set of measure* 0, *will lie in a region v; that is,*

$$\lim_{t = \pm \infty} \frac{\bar{t}}{t} = p \leqq 1$$

*will exist, where t denotes total elapsed time measured from a fixed point and* $\bar{t}$ *the elapsed time in v.*

For a strongly transitive system $p$ is, of course, the ratio of the volume of $v$ to $V$.

Evidently the germ of the above argument is contained in the lemma. The abstract character of this lemma is to be observed, for it shows that the theorem above will extend at once to function space under suitable restrictions.

It is obvious that $\tau(P)$ and $\bar{\tau}(P)$ as defined above satisfy functional relations of the following type:

$$\int_0^\lambda \lambda dm(S_\lambda) = \int_{S_\lambda} t(P) dP$$

where the integral on the left is a Stieltjes integral, $m(S_\lambda)$ being the measure of $S_\lambda$.

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## *ERRATA*

## *CORRECTION TO "A SET OF AXIOMS FOR DIFFERENTIAL GEOMETRY"*

BY O. VEBLEN AND J. H. C. WHITEHEAD

In the last sentence but one of § 9 we stated that the union of all normal coördinate systems at a point $Q$, for a given coördinate system $P \longrightarrow x$, exists. This is not necessarily true, as may be seen by considering normal coördinate systems for a cylinder with a locally Euclidean metric.

* These PROCEEDINGS, **17**, 551–562 (1931).

# SUMMARY OF RESULTS AND PROOFS ON FERMAT'S LAST THEOREM (SIXTH PAPER)

BY H. S. VANDIVER

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In a recent paper[1] the writer proved a number of results concerning Fermat's Last Theorem. In another article[2] he gave a number of sidelights and comments on the contents of that paper, as well as extensions of the theorems therein. In the present paper further results will be given along these lines.

As noted in $N$, Theorem IV of $T$ is quite complicated and in the former paper another proof is indicated. I give here, however, a method for proving the statement which is far simpler than either of those just referred to. We shall assume the relation (27) of $T$, that is

$$\omega + \zeta\theta \equiv 0 \pmod{\mathfrak{P}} \tag{1}$$

omitting the superscripts in $\omega$ and $\theta$. Hence the relation (27a) of $T$ takes the form

$$\begin{gathered}(\omega + \zeta^a\theta)^c \equiv (\zeta^a\omega + \theta)^c \pmod{\mathfrak{P}} \\ a = 0, 2, 3, \ldots, l-2.\end{gathered} \tag{2}$$

$$\text{Set } f(a) = (\omega + \zeta^a\theta)^c, \text{ and } g(a) = (\zeta^a\omega + \theta)^c.$$

Expansion of (2) gives

$$\begin{gathered}\omega^c + c\omega^{c-1}\zeta^a\theta + \binom{c}{2}\omega^{c-2}\zeta^{2a}\theta^2 + \ldots + \zeta^{ac}\theta^c \equiv \\ \zeta^{ac}\omega^c + c\zeta^{a(c-1)}\omega^{c-1}\theta + \binom{c}{2}\zeta^{a(c-2)}\omega^{c-2}\theta^2 + \ldots\end{gathered} \tag{3}$$

As noted before this is true for $a = 0, 2, 3, \ldots, l-2$. Noting that $f(1) \equiv g(-1) \equiv 0 \pmod{\mathfrak{P}}$ we then obtain

$$-f(-1) + \sum_{i=0}^{l-1} f(i) \equiv -g(1) + \sum_{i=0}^{l-1} g(i) \pmod{\mathfrak{P}}.$$

Using $(\zeta^k)^{l-1} + (\zeta^k)^{l-2} \ldots + 1 = 0$ for $k \not\equiv 0 \pmod{l}$, and noting that $c < l$, this gives

$$l\omega^c - (\omega + \zeta^{-1}\theta)^c \equiv l\theta^c - (\zeta\omega + \theta)^c \pmod{\mathfrak{P}},$$

and using $\omega \equiv -\zeta\theta \pmod{\mathfrak{P}}$ this reduces to

$$-\zeta^c l \equiv (1 - \zeta^2)^c \pmod{\mathfrak{P}}. \tag{4}$$

In the same way we have

$$-\zeta f(-1) + \sum_{i=0}^{l-1} \zeta^{-i} f(i)$$
$$\equiv -\zeta^{-1} g(1) + \sum_{i=0}^{l-1} \zeta^{-i} g(i) \pmod{\mathfrak{P}}$$

and this gives $lc\zeta^c(\zeta^{c-2} - 1) \equiv (\zeta^{c-2} - 1)(1 - \zeta^2)^c$.

Now if $c \neq 2$ we have $lc \equiv (1 - \zeta^2)^c \pmod{\mathfrak{P}}$ and comparison with (4) gives $c \equiv -1 \pmod{\mathfrak{P}}$, which is impossible.

For $c = 2$, we obtain from (1) and (2)

$$(\zeta - \zeta^a)^2 \equiv (\zeta^{a+1} - 1)^2.$$

Set $a = 2$, this gives

$$\zeta - \zeta^2 \equiv \pm(\zeta^3 - 1) \pmod{\mathfrak{P}}, \tag{5}$$

and the plus sign gives

$$\zeta^3 + \zeta^2 \equiv \zeta + 1$$

and

$$(\zeta^2 - 1)(\zeta + 1) \equiv 0 \pmod{\mathfrak{P}},$$

which is impossible, since $(\zeta + 1)$ is a unit and $(\zeta^2 - 1)$ is a prime ideal factor of $(l)$, which is prime to $p$. The minus sign in (5) gives

$$(\zeta^2 + 1)(\zeta - 1) \equiv 0 \pmod{\mathfrak{P}},$$

which is likewise impossible.

Hence we have proved that $\omega + \theta$ is divisible by $p$ in (26a) of $T$.

The above proof applies in connection with Theorem IV but not in connection with the proof of Theorem V of $T$, since in the latter case the $p$ we are using is not necessarily less than $(l^2 - l)$. The argument for the proof of $\theta + \omega = 0 \pmod{p}$ which was used in $T$ in connection with the proof of Theorem IV is necessary to supply a similar step in the proof of Theorem V of $T$, since that part of the argument in the proof of Theorem IV does not depend upon the fact that $p$ is less than $(l^2 - l)$.

We now consider the relation

$$\alpha^l + \beta^l + \eta\gamma^l = 0 \tag{6}$$

where $l$ is a regular prime; $\alpha$, $\beta$ and $\gamma$ are integers in the field $k(\zeta)$ none zero, and $\eta$ is a given unit in this field. Let us assume first that $\gamma$ is prime to $\lambda = (1 - \zeta)$. Since we may take $\alpha$ and $\beta$ each in the form $f \pmod{\lambda^2}$, it follows that $\alpha^l \equiv a \pmod{\lambda^l}$, $\beta^l \equiv b \pmod{\lambda^l}$ and $\gamma^l \equiv c \pmod{\lambda^l}$, where $a$, $b$ and $c$ are rational integers. Reducing our equation $\pmod{\lambda^l}$ we obtain,

$$a^l + b^l + \eta c^l \equiv 0 \pmod{\lambda^l}.$$

Since $\gamma$ is prime to $\lambda$ it follows that

$$c \not\equiv 0 \pmod{\lambda},$$

hence we have,

$$\eta \equiv -\frac{a^l + b^l}{c^l} \pmod{\lambda^l}$$

Hence $\eta$ is congruent to a rational integer, (mod $\lambda^l$), and hence is primary

Since the field is regular, however,[3] it follows by a known result that $\eta$ is the $l$th power of a unit in $k(\zeta)$. Hence our relation (6) becomes

$$\alpha^l + \beta^l + \gamma_1^l = 0 \tag{7}$$

where $\gamma_1$ is an integer in $k(\zeta)$. This relation is known[4] to be impossible, hence (6) is impossible for $\gamma$ prime to $\lambda$.

Consider now the case where $\gamma$ is divisible by $\lambda$ in (6). In both the known[4] proofs that

$$\alpha^l + \beta^l + \gamma^l = 0$$

is impossible where $\gamma$ is divisible by $\lambda$; the arguments include a proof that (6) is impossible for $\gamma$ divisible by $\lambda$. For the first steps in these arguments take the equation in the general form (6). Hence we may state the

THEOREM. *The equation*

$$\alpha^l + \beta^l + \eta\gamma^l = 0$$

*is impossible for $\alpha$, $\beta$ and $\gamma$ integers in the field $k(\zeta)$ none zero where $\eta$ is a given unit in this field and $l$ is a regular prime.*

So far in connection with known proofs of Fermat's Last Theorem for regular primes the discussion has been divided into two quite distinct parts. The first part is confined to the case where $x$, $y$ and $z$ are prime to each other and $xyz \not\equiv 0 \pmod{l}$, called case I, in the relation

$$x^l + y^l + z^l = 0. \tag{7a}$$

The second part of the proof considers the case where one of the integers is divisible by $l$, called case II. The treatments of case II have always involved some form of Fermat's famous method of infinite descent. The writer has constructed a proof that (6) is impossible in the special case where $\alpha$, $\beta$ and $\gamma$ belong to the field $\Omega(\zeta + \zeta^{-1})$ prime to each other, in which the treatment of case I, that is when $\alpha$, $\beta$ and $\gamma$ are prime to $\lambda$, also involves the method of infinite descent. In fact, the argument is so

constructed that the two cases are treated simultaneously during the descent. The equation is taken in the form

$$\theta^l + \omega^l + \delta\gamma^l = 0. \tag{7b}$$

For the case $\gamma$ prime to $\lambda$ we have

$$\theta + \omega\zeta^a = \eta_a^l \sigma_a^l \; ; \qquad a = 0, 1, \ldots, l-1.$$

We deduce from this by known methods the relation $\theta \equiv \omega \pmod{\lambda^l}$ and this gives

$$\theta + \omega\zeta^a = (1 + \zeta^a)\eta_a\sigma_a^l$$

where $\eta_a$ is a real unit in $k(\zeta)$.

Also by known methods we have for the case where $\gamma$ is divisible by $\lambda$ the relation

$$\frac{\theta + \zeta^a\omega}{1 - \zeta^a} = \eta''_a\rho_a^l.$$

The last two relations may be combined into the statement

$$\frac{\theta + \omega\zeta^a}{1 \pm \zeta^a} = \eta_a\sigma_a^l \; ; \qquad a = 1, 2, \ldots, l-1. \tag{8}$$

From this relation we obtain

$$\sigma_{-a}^{\,l}\,\frac{\theta + \omega\zeta^a}{1 \pm \zeta^a} = \frac{\theta + \omega\zeta^{-a}}{1 \pm \zeta^{-a}}\,\sigma_a^{\,l}$$

where $\sigma_{-a}$ is obtained from $\sigma_a$ by the substitution $\zeta/\zeta^{-1}$. This is considered (mod $\mathfrak{P}$) where $\mathfrak{P}$ is a prime ideal divisor of $\theta + \omega\zeta$. Using power characters we infer as in a former paper by the writer[4a]

$$\left\{\frac{E_n}{\mathfrak{P}}\right\} = 1; \quad n = 1, 2, \ldots, \frac{l-3}{2}.$$

Hence

$$\left\{\frac{E_n}{\sigma_1}\right\} = 1,$$

where

$$E_n = \prod_{i=0}^{(l-3)/2} \epsilon(\zeta^{ri})^{r^{-2in}}$$

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1})}\right)^{1/2},$$

$r$ being a primitive root of $l$. By taking the known expressions for power

characters of units in a cyclotomic field we find after some transformations, the relation

$$\sigma_1 \equiv \sigma_{-1} \pmod{\lambda^{l-1}}.$$

Using (8) we have

$$\theta + \omega\zeta = (1 \pm \zeta)\eta_1\sigma_1^l$$
$$\theta + \omega\zeta^{-1} = (1 \pm \zeta^{-1})\eta_{-1}\sigma_{-1}^l$$
$$\lambda^t(\theta + \omega) = \eta_0\sigma_0^l$$

where the exponent $t = 0$ or $(-1)$, according as $\gamma$ is not or is divisible by $\lambda$. Eliminating $\theta$ and $\omega$ from the last three equations we obtain

$$\sigma_1^l \pm \sigma_{-1}^l = \eta_0^1\sigma_0^l$$

where $\eta_0^1$ is a unit in $k(\zeta)$. This leads to

$$\sigma_1 \pm \zeta^a\sigma_{-1} = (1 \pm \zeta^a)\xi_a\tau_a^l, \tag{9}$$

the ambiguous sign being positive or negative according as $\gamma$ is not or is divisible by $\lambda$. Using the relation obtained from (9) by substituting $(-a)$ for $a$ throughout and employing $\sigma_a \equiv \sigma_{-a} \pmod{\lambda^{l-1}}$, we find that $\xi_a/\xi_{-a}$ is primary, and hence, since the field is regular, it is the $l$th power of a unit in said field. Taking $a = 1, -1$, in (9) together with

$$\lambda^t(\sigma_1 + \sigma_{-1}) = \xi_0^1\tau_0^l,$$

and eliminating $\sigma_1$ and $\sigma_{-1}$ from the three resulting equations we have

$$\tau_1^l + \frac{\xi_{-1}}{\xi_1}\tau_{-1}^l + \delta_1\tau_0^l = 0.$$

Using the fact that $\xi_{-1}/\xi$ is an $l$th power we obtain

$$\theta_1^l + \omega_1^l + \delta_1\gamma_1^l = 0, \tag{10}$$

which is the same form as (7b), since $\theta_1$, $\omega_1$ and $\gamma_1$ each belong to $\Omega(\zeta + \zeta^{-1})$ and are prime to each other. We may now employ the same transformations on (10) as were used in connection with (7b) and we shall obtain the relation

$$\theta_2^l + \omega_2^l + \delta_2\gamma_2^l = 0.$$

Proceeding in this way we find that we get an infinite series of equations of this type with the successive $\gamma$'s each containing less ideal prime factors than the preceding, which is impossible unless a certain $\gamma$ is a unit in $k(\zeta)$. But this is easily shown to lead to a contradiction.

An extension of this method leads to the proof of the impossibility of (7b) where $\delta$ is replaced by given types of integers in $k(\zeta)$.

*Determination of Some Properly Irregular Cyclotomic Fields.*—The integer $h$ which represents the number of classes of ideals in a cyclotomic field defined by $e^{2i\pi/l}$, where $l$ is an odd prime, can be written in the form $h_1h_2$, where $h_1$ and $h_2$ are both integers.[10] If $h \equiv 0 \pmod{l}$ the field is called irregular. The necessary and sufficient condition that $h_1$ be divisible by $l$ is that one of the first $(l-3)/2$ Bernoulli numbers be divisible by $l$. A necessary, but not a sufficient, condition that $h_2$ be divisible by $l$ is that $h_1$ be divisible by $l$. A cyclotomic field in which $h_1$ is divisible by $l$, but $h_2$ is prime to $l$, is called a *properly irregular cyclotomic field.*

In a former paper I showed that this definition of a properly irregular cyclotomic field is equivalent to the statement[5] that none of the units $E_n$ already defined, are $l$th powers of units in $k(\zeta)$. In another paper by Miss Elizabeth T. Stafford and myself[6] it was found that all the irregular fields, defined by primes $l < 211$, were also properly irregular.

In order to determine if a given cyclotomic field is properly irregular for an $l > 210$, it is first necessary to test the same as to regularity. As noted in another article,[7] this test was made by Miss E. M. Badger, for all primes $l$; $210 < l < 269$. All were found regular between these limits except 233, 257 and 263; for each $l$ the only $B$'s divisible by $l$ in the set

$$B_1, B_2, \ldots, B_{(l-3)/2} \tag{11}$$

are as follows:

$$233, B_{42}; \quad 257, B_{82}; \quad 263, B_{50}$$

These tests were continued by Mr. M. M. Abernathy who examined the regularity of all primes $l$; $268 < l < 307$, this work being included in his M.A. thesis at the University of Texas. All were found regular within the limits just mentioned except 271, 283 and 293. The Bernoulli numbers were congruent to 0 (mod $l$) as follows:

$$271, B_{42}; \quad 283, B_{10}; \quad 293, B_{78}.$$

On page 145 of a former article[6] a method was described for testing the regularity of the prime $l = 127$. We depended primarily on the formula

$$\sum_{s=[l/6]+1}^{[l/4]} s^{2a-1} \equiv \frac{1 - 3^{l-2a} - 4^{l-2a} + 6^{l-2a}}{4a} (-1)^a B_a \pmod{l},$$

where $2a < (l-1)$ and [x] denotes the greatest integer in $x$. For a particular $s$ the expressions $s^{2a-1}$ were computed for the successive values starting with $a = 12$. The tests of Mr. Abernathy referred to above were the same as just indicated, except, in order to apply certain checks on the accuracy of the computations, the columns were included corresponding to all the integers $a = 1, 2, \ldots, \frac{l-1}{2}$. It was then easy to

show that the sum of the integers that were congruent to $s^{2a-1}$ for a particular $s$, gives an integer divisible by $l$. Also, after the totals of all columns were made, the sum of these totals must be congruent to zero.[7a] For the cases 271 and 283 the corresponding Bernoulli numbers were found in Adams' *Tables*, and the numerators divided by the respective primes gave remainders of zero in each case.

Having determined that the primes 233, 257, 263, 271, 283 and 293 are all irregular, each one was tested as to being properly irregular, by considering the unit $E_n$ and the possibility of its being the $l$th power of a unit in $k(\zeta)$ for the values of $n$ corresponding to the subscripts of the Bernoulli numbers divisible by $l$ in each case. The method used for the irregular primes $< 211$ is described on page 148 of another article.[6] The method employed for the primes now under discussion was the same, aside from the fact that these computations were carried out much more systematically and a number of new devices for shortening the calculations were used. If

$$E_i = \delta^l \tag{11a}$$

where $\delta$ is an integer in $k(\zeta)$ let $p$ be the smallest rational prime, such that $p \equiv 1 \pmod{l}$. Then $p$ decomposes in $k(\zeta)$ into the product of $(l-1)$ distinct ideal factors. Let $d$ be an integer such that $d^l \equiv 1 \pmod{p}$. Then one of the ideal factors mentioned is $\mathfrak{P} = (\zeta - d, p)$, which gives $\zeta \equiv d \pmod{\mathfrak{P}}$. Consider the expression (11a) as a congruence (mod $\mathfrak{P}$); in view of the fact that $\zeta \equiv d \pmod{\mathfrak{P}}$ we obtain

$$E_n(d) \equiv c^l \pmod{p}$$

where $c$ is a rational integer. Our problem is now reduced to determining if the index for $E_n(d)$ is divisible by $l$ if we use a table of indices for the modulus $p$. In carrying this out, $E_n(d)$ was written in the following form

$$d^R \prod_{i=0}^{(l-3)/2} \left(\frac{d^{r^{i+1}} - 1}{d^{r^i} - 1}\right)^{r^{(l-1-2n)i}}$$

$$R = \frac{1-r}{2}\left(1 + r^h + r^{2h} + \ldots + r^{\frac{l-3}{2}h}\right)$$

$$h = l - 2n.$$

The exponent $R$ in the first factor reduces immediately, modulo $l$, to

$$\frac{r-1}{r^{l-2n}-1}.$$

The computation of the index of $d^R$ involves an obvious procedure, we now consider the determination of the index of the other factor, which

factor we shall call $F$. We shall describe the major part of the procedure in some detail as it is a bit elaborate. As an example take the case $l = 271$. Here $p = 1627$ and we take for the primitive root $r$ of 271, the value 269 so that we may employ Jacobi's tables of indices for the prime 271. Also $n = 42$. A partial table of indices for the prime 1627 was then constructed by taking the primitive root 3 and finding the least positive residues of $3^k$; $k = 1, 2, \ldots, 271$. A companion table giving the indices corresponding to all the least residues just mentioned, with blank spaces left for the indices of numbers not appearing in this set, was also made. Owing to the fact that only the indices reduced modulo $l$ were required for the numbers involved in $F$, the partial tables just described readily yielded this information. For the prime being considered the integer $d$ was taken as $3^6$; the first ten rows and columns giving the computations in connection with our prime are exhibited below:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $r^i$ | 1 | 269 | 4 | 263 | 16 | 239 | 64 | 143 | 256 | 30 |
| $6r^i$ | 6 | 1614 | 24 | 1578 | 96 | 1434 | 384 | 858 | 1536 | 180 |
| $d^{r^i} - 1$ | 728 | 284 | 819 | 981 | 1479 | 1253 | 1507 | 826 | 1289 | 369 |
| $\text{ind}(d^{r^i} - 1)$ | 39 | 78 | 167 | 154 | 146 | 13 | 138 | 26 | 193 | 118 |
| $D(i)$ | 39 | 89 | 258 | 263 | 138 | 125 | 159 | 167 | 196 | 102 |
| $r^{186i}$ | 1 | 166 | 185 | 87 | 79 | 106 | 252 | 98 | 8 | 244 |
| $186i$ | 0 | 186 | 102 | 18 | 204 | 120 | 36 | 222 | 138 | 54 |
| $K(i)$ | 39 | 140 | 34 | 117 | 62 | 242 | 231 | 106 | 213 | 227 |

Here

$$D(i) = \text{ind}\left[\frac{d^{r^{i+1}} - 1}{d^{r^i} - 1}\right]$$

$$K(i) = \text{ind}\left[\frac{d^{r^{i+1}} - 1}{d^{r^{i+1}} - 1}\right]^{r^{186i}}$$

The rows and columns are continued up to $i = 134$ inclusive. The elements of the second row are obtained from the tables of indices modulo 271. The third row is obtained immediately from the second and the multiplier 6 is used owing to the fact that $d = 3^6$. The elements of the fourth row are obtained from the partial table of indices modulo 1627, as well as those of the fifth row. The number in the sixth row are obtained by an obvious subtraction of certain elements in the preceding row. The elements in the eighth row are obtained by reducing $186i$ modulo 270, and from these numbers the data in the seventh row is obtained using a table of indices for the prime 271. The corresponding elements in the sixth and seventh rows are then multiplied together and reduced modulo 271 to give the numbers in the last row, which are then added together and this total is in turn added to the index for $d^R$ to obtain the residue when the index of $E_n(d)$ is reduced modulo 271.

Checks were employed on the accuracy of the computations for each row excepting the fourth. Summing all the numbers in the first row and using the fact that they formed a geometric progression, the result should be congruent to

$$-\frac{2}{r-1} \pmod{271}.$$

Similarly we have obvious checks on the second, fifth, sixth and seventh rows. No check was employed on the fourth row but the accuracy of the numbers therein was involved in the check employed for the fifth row, which was more elaborate than the other checks referred to, and depends on the following transformations:

We have

$$(\zeta - 1)(\zeta^r - 1)(\zeta^{r^2} - 1) \ldots (\zeta^{r^{l-2}} - 1) = l$$

$$A = (\zeta - 1)(\zeta^r - 1)(\zeta^{r^2} - 1) \ldots (\zeta^{r^{\frac{l-3}{2}}} - 1), \qquad (12)$$

then

$$AB = l,$$

where

$$B = (\zeta^{-1} - 1)(\zeta^{-r} - 1)(\zeta^{-r^2} - 1) \ldots (\zeta^{-r^{\frac{l-3}{2}}} - 1),$$

whence

$$B = (-1)^{\frac{l-1}{2}} \zeta^h A,$$

where $h = -1 - r \ldots - r^{\frac{l-3}{2}}$.

Now

$$h \equiv -\frac{r^{\frac{l-1}{2}} - 1}{r-1} \equiv \frac{2}{r-1} \pmod{l},$$

hence (12) gives

$$\zeta^{\frac{2}{r-1}}(-1)^{\frac{l-1}{2}} A^2 = l. \qquad (13)$$

In $A$ set $d$ in place of $\zeta$. Denote the result by $A(d)$. Then using $\zeta \equiv d \pmod{\mathfrak{P}}$ we have from (13), by taking indices

$$\left[\frac{2}{r-1}\right] \text{ind } d + \frac{l-1}{2} \cdot \frac{p-1}{2} + 2 \text{ ind } A(d) \equiv \text{ind } l \pmod{(p-1)}$$

where the symbol

$$\left[\frac{2}{r-1}\right]$$

denotes an integer $i$ such that $i(r-1) \equiv 2 \pmod{l}$, and this formula furnishes the check.

The table below gives the values of $n$, $r$, $d$, $p$ and ? for each irregular prime $l$; $210 < l < 307$. The last column headed ind $E_n(d)$ gives the residue of this integer mod $l$ in each case.

| $l$ | $n$ | $r$ | $d$ | $p$ | $\rho$ | Ind $E_n(d)$ |
|---|---|---|---|---|---|---|
| 233 | 42 | 10 | 100 | 467 | 10 | 13 |
| 257 | 82 | 10 | 136 | 1543 | 10 | 123 |
| 263 | 50 | 10 | 729 | 1579 | 3 | 171 |
| 271 | 42 | 269 | 729 | 1627 | 3 | 4 |
| 283 | 10 | 273 | 729 | 1699 | 3 | 136 |
| 293 | 78 | 204 | 100 | 587 | 577 | 291 |

As the table shows, the computation established the fact that $E_n(d)$ is not divisible by $l$, and hence $E_n(\zeta)$ is not the $l$th power of a unit in the filed $k(\zeta)$ for any of the primes mentioned. It follows that the cyclotomic fields defined by each of these primes are properly irregular cyclotomic fields.

All the calculations of the type just mentioned were carried out by Mr. M. E. Tittle, Miss B. Bennett and Mr. M. M. Abernathy, with the aid of funds provided by the University of Texas. In particular, Mr. Tittle discovered some of the devices employed for shortening the computations.

*Application of Properly Irregular Cyclotomic Fields to Fermat's Last Theorem.*—In $T$ I proved a number of theorems concerning Fermat's last theorem. In the present paper I shall consider the application to special exponents in the Fermat relation of the following theorems included in $T$ (numbered as in $T$).

THEOREM I. *Under the following assumptions:*

(1) *the second factor of the class number of the field $k(\zeta)$ is prime to $l$;*

(2) *none of the Bernoulli numbers $B_n$, $n = 1, 2, \ldots, (l-3)/2$, is divisible by $l^3$;*

*the equation* (7a) *is impossible in case II.*

THEOREM III. *If $l \equiv 1 \pmod 4$ and all the numbers in* (11) *which are divisible by $l$ have even subscripts, then* (7a) *is impossible in rational integers none zero, provided*[8] *also that the second factor of the class number of $k(\zeta)$ is prime to $l$.*

THEOREM IV. *Under the following assumption:*

*None of the units $E_a$, $a = a_1, a_2, \ldots, a_s$, is congruent to the $l$th power of an integer in the field $k(\zeta)$ mod $\mathfrak{P}$, where $\mathfrak{P}$ is a prime ideal divisor of $p$, $p$ is a prime $< (l^2 - l)$ of the form 1 mod $l$, and $a_1, a_2, \ldots, a_s$ are the subscripts in the Bernoulli numbers in the set* (11) *which are divisible by $l$;*

*the relation* (7a) *is impossible in case II.*

THEOREM V. *Under the following assumptions:*

(1) *there exists a rational prime integer $p$ such that the congruence*

$$u^l + v^l + w^l \equiv 0 \pmod{p}$$

*has no solution $u$, $v$ and $w$ all rational integers prime to $p$, and $p \not\equiv 1 \pmod{l^2}$;*

(2) *the relation*

$$\left\{\frac{E_a}{\mathfrak{P}}\right\} \neq 1$$

*holds, where $a$ ranges over the values $a_1, a_2, \ldots, a_s$, these integers being the subscripts of Bernoulli numbers in the set* (11) *which are divisible by $l$, and $\mathfrak{P}$ is a prime ideal divisor of $p$;*

*the equation* (7a) *is impossible in rational integers none zero.*

In Theorem V the symbol

$$\left\{\frac{E_n}{\mathfrak{P}}\right\}$$

denotes the $l$th power-character of $E_n$ with respect to the ideal $\mathfrak{P}$.

The proofs of the above-mentioned theorems were more or less different. In particular, if we substitute the argument given at the beginning of the present paper in lieu of the argument given on pages 632–635 for the proof that $\omega + \theta \equiv 0 \pmod{p}$, the proof of Theorem IV is largely different from that of Theorem V if we retain the original argument as to $\omega + \theta \equiv 0 \pmod{p}$ as part of the proof of Theorem V, as is necessary, since in the latter case we cannot assume $p = 1 + cl$ with $c < (l - 1)$.

In $T$, the first three theorems quoted above were applied to the particular values of $l < 211$, which were irregular. Here I shall consider the application of these three theorems to (7a) for $210 < l < 307$, and irregular; and the application of Theorem V to all irregular values of $l < 307$.

Concerning Theorem I the first assumption holds for all irregular primes $l$, $210 < l < 307$, since we have shown that all the cyclotomic fields defined by these $l$'s are properly irregular. As to the second assumption in Theorem I, this was shown to hold for the cases $l = 233, 257$ and $263$, as stated in $N$, page 303. This assumption was examined by Mr. M. M. Abernathy for the case $l = 271$, who found

$$A_{42} \equiv 271 \cdot 181 \pmod{271^2},$$

where

$$A_n = \frac{(-1)^n B_{nl}(2^{2nl} - 1)}{2^{2nl}\, nl}.$$

The case $l = 283$ was disposed of by Mr. J. A. Clack, who found,

$$A_{10} \equiv 283 \cdot 71 (\text{mod } 283^2) .$$

Hence Theorem I proves the last theorem for all irregular $l$'s, $210 < l < 307$, excepting 293, for which latter prime it was not tested.

The first two assumptions mentioned in Theorem III are satisfied only for $l = 233$, 257 and 293, within the range mentioned. The third assumption mentioned holds as already shown. Hence Theorem III proves the last theorem for $l = 233$, 257 and 293.

As to the application of Theorem IV, the results of the computations in establishing the fact that the irregular cyclotomic fields defined by the irregular primes $l$, $210 < l < 307$ are properly irregular, incidentally show that the assumptions in Theorem IV holds for all these primes, since the value of $p$ selected was $< (l^2 - l)$ for each $l$. Hence Theorem IV proves the last theorem for all irregular $l$'s such that $210 < l < 307$.

The application of Theorem V to special exponents has not been mentioned in any of my preceding papers. The first assumption in it states that there exists a rational prime integer $p$ such that the congruence

$$u^l + v^l + w^l \equiv 0 (\text{mod } p) \qquad (14)$$

has no solution $u$, $v$ and $w$ all rational integers prime to $p$, and $p \not\equiv 1 (\text{mod } l^2)$. The congruence mentioned has obvious solutions when $p \equiv 1 (\text{mod } 3)$. Using the results of Dickson,[9] we note that the congruence (14) is impossible for $l = 37$, $p = 149$, hence the first assumption of Theorem V holds for $l = 37$, the second assumption holds, since 149 is the value of $p$ used in verifying that

$$\left\{\frac{E_{16}}{\mathfrak{P}}\right\} \not\equiv 1$$

where $l = 37$. In a similar way our computations concerning the properly irregular cyclotomic fields mentioned in $T$, page 641–642, and in the present paper, together with Dickson's results concerning the trinomial congruence, proved the last theorem, for

$$l = 37, 67, 101, 131, 149, 157, 233 \text{ and } 293.$$

For the irregular primes $< 307$ not included in the list just mentioned, the values of $p$ which were used in our irregular field computations are in each case of the form $6l + 1$; consequently the congruence (14) has solutions. Hence, in order to test Theorem V for the exceptional values of $l$ such as 59, 103, etc., it would be necessary to select a value of $p$ such that $p \not\equiv 1 (\text{mod } 3)$ and not included in Dickson's exceptions. We have not carried out such computations and the questions as to whether Theorem

V yields proofs of the last theorem for all irregular primes $< 307$ remains open.

In $T$, page 614 and elsewhere,[10] the writer has referred to Kummer's results on Fermat's last theorem for irregular prime exponents. The first assumption of the theorem states that the first factor of the class number of $k(\zeta)$ is divisible by $l$ but not by $l^2$. Hence, under this restriction Kummer's results may be applied to special exponents in the Fermat relation, since his assumptions II and III are equivalent to those of Theorem I of $T$. However, owing to assumption I, Kummer's argument, as corrected by the writer,[11] is far simpler than the proof given of Theorem I in $T$; the proof of the latter requiring lemma I of $T$ which involves in its proof Furtwängler's law of reciprocity. Also the proof of the lemma depending upon the results of Takagi involves the existence of a class-field. Kummer's proof depends on considerations much simpler than those involved in the proof of the existence of a class-field or of the law of reciprocity. Also, in other respects Kummer's argument is different from that I have given on pages 621–624 of $T$. Hence we may regard his work as furnishing more or less different proofs of the last theorem for all irregular $l$'s $< 307$, excepting 157, in which case the first factor of the class number of $k(\zeta)$ is divisible by $157^2$.

To summarize, we shall now list the irregular primes $< 307$, together with the number of different proofs, which have been mentioned in this paper, in each case:

37,5; 59,3; 67,3; 101,5; 103,3; 131,4; 149,4; 157,3; 233,5; 257,4; 263,3; 271,3; 283,3; 293,4.

Using also the fact that Fermat's last theorem is true for regular prime exponents, we may then state that the last theorem has been proved for all exponents $< 307$.

[1] *Trans. Am. Math. Soc.*, **31**, 613–642 (1929). This paper will be referred to as $T$.

[2] These PROCEEDINGS, **16**, 298–305 (1930). This paper will be referred to as $N$.

[3] Landau, *Vorlesungen über Zahlentheorie*, **3**, 240.

[4] Kummer, *Crelle*, **40**, 130–138 (1850); Landau, loc. cit., 271–274.

[4a] *Trans. Am. Math. Soc.*, **31**, 633 (1929).

[5] These PROCEEDINGS, **16**, 743–749 (1930).

[6] *Ibid.*, **16**, 139–150 (1930).

[7] *Ibid.*, **16**, 303 (1930).

[7a] Miss Stafford's previous computations concerning the regularity of the primes $l$ such that $157 < l < 211$, were checked by Mr. Abernathy by the methods just described concerning the sums of the rows and columns, and Miss Stafford's conclusion that all such primes are regular, was confirmed.

[8] Cf. errata to Vol. **33** of the *Trans. Am. Math. Soc.*

[9] *Messenger of Math.*, [2], **38**, 14–32 (1908).

[10] *Bull. Nat. Res. Coun.*, **62**, Feb., 1928, 34, 44.

[11] *Bull. Am. Math. Soc.*, **28**, 400–407 (1922); these PROCEEDINGS, **12**, 767–772 (1926).

# PROPERTIES OF GROUP MANIFOLDS

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Let $M$ be an $n$-dimensional topological manifold, compact or not and suppose that there can be defined over $M$ a correspondence $T$ which associates in a continuous manner a single point $Z$ to every ordered pair $X$, $Y$, — in symbols, $XY = Z$. Suppose further that for one point $E$, $XE = EX = X$ for every point $X$.

*The Fundamental Group* $\Gamma$ *of* $M$ *Is Abelian.*—This theorem was proved by Schreier[1] for the special case in which $T$ defines a continuous group. A simple proof is the following: let $\alpha$ and $\beta$ be two elements of $\Gamma$; they are oriented closed curves defined by the equations $X = \alpha(s)$, $X = \beta(t)$, where $s$, $t$ are points ranging over the oriented circles $\sigma$, $\tau$, respectively. We may suppose that $\alpha(s_0) = \beta(t_0) = E$. Now the torus $\sigma \times \tau$ is mapped on $M$ by the continuous single-valued correspondence $\xi$ defined as follows: to the arbitrary point $s \times t$ of $\sigma \times \tau$, there is made to correspond the point $\alpha(s)\ \beta(t)$ of $M$. Under $\xi$, the images of the oriented curves $a = s_0 \times \tau$ and $b = \sigma \times t_0$ are $\alpha$ and $\beta$, respectively. On $\sigma \times \tau$ the curve $aba^{-1}b^{-1}$ is deformable to a point. Under $\xi$ this deformation is imaged by a deformation of $\alpha\beta\alpha^{-1}\beta^{-1}$ to a point. Hence $\alpha$ and $\beta$ are permutable, and $\Gamma$ is abelian.

We shall now assume that $(XY)Z = X(YZ)$ for arbitrary points $X$, $Y$, $Z$ of $M$. Then *there exists a uniquely determined* $n$*-dimensional submanifold* $M_1$ *of* $M$ *over which* $T$ *defines an* $n$*-parameter continuous group* $G$. $M$ is the group (or parameter) manifold of $G$. We shall outline the proof. Let $U$ be a neighborhood of $E$ which is an $n$-cell. The set of points $XU$, determines, for a fixed $X$, a singular image of $U$ on $M$, and if $X$ lies in a suitably restricted neighborhood $V$ of $E$, the points of $XU$ will be uniformly near the corresponding points of $U$. By simple continuity considerations, it follows that $XU$ covers $E$ whenever $X$ is in $V$. Hence for $X$ in $V$, there is at least one $X'$ in $U$ such that $XX' = E$—that is, each $X$ in $V$ has a "right-inverse" in $U$. We can in the same way choose a neighborhood $W$ of $E$ such that each point of $W$ has a right-inverse in $V$. Let $X$ be in $W$, $X'$ a right-inverse of $X$ in $V$, and $X''$ a right inverse of $X'$. From $(XX')X'' = X(X'X'')$, we have $EX' = XE$ or $X'' = X$, so that $X'X = E$. Hence if $XY = E$, we have $Y = X'$. Hence each point of $W$ has a true and unique inverse. It now follows readily that the set $W^{-1}$ consisting of the inverses of the points of $W$, contains $E$ and is homeomorphic with $W$. Thus the set $W_1 = W + W^{-1}$ is a connected open set containing $E$, and such that $W_1^{-1} = W_1$. The same is true of

the set $W_1 W_1 \ldots W_1$, hence of the set $M_1 = W_1 + W_1 W_1 + \ldots$ and since $M_1 M_1 = M_1$, it follows that $T$ defines a group over $M_1$.

The proof that $M_1$ is unique offers no difficulty.

A simple example shows that $M_1$ need not be identical with $M$. A necessary and sufficient condition that $M_1$ be identical with $M$ is that if $X_n \longrightarrow X$ and $X_n Y_n = E$, then the sequence $Y_n$ has at least one converging subsequence. This condition is automatically satisfied when $M$ is compact.

Suppose now that $M$ is the group manifold of a group $G$. From the preceding results we deduce immediately that if $N$ is a $k$-dimensional sub-manifold of $M$ containing $E$ and with the property $NN = N$, then $G$ defines a group over $N$—namely, a $k$-parameter subgroup of $G$. In particular if $N$ is a simple closed curve $j$, then $j$ is the group manifold of a one-parameter subgroup $g$ of $G$. Such subgroups are of frequent occurrence. We shall say that a number of them $g_1, g_2, \ldots, g_p$ are independent if their group manifolds $j_1, j_2, \ldots, j_p$, when oriented and regarded as elements of $\Gamma$, satisfy no relation of the type $j_1^{\alpha_1} j_2^{\alpha_2} \ldots j_p^{\alpha_p} = 1$. Concerning the structure of abelian groups, we have the theorem that *an n-parameter abelian group admits at most n independent one-parameter subgroups*. The proof consists in showing that if $j_1, j_2, \ldots, j_{n+1}$ are manifolds of independent one-parameter subgroups, then the product manifold $j_1 \times j_2 \times \ldots j_{n+1}$ is mapped on $M$ in a continuous non-singular manner, at least in the neighborhood of $E$; this contradicts the invariance of dimensionality.

We remark finally that a *group manifold M is orientable*. For suppose that an oriented $n$-cell $H$ can be isotopically deformed on $M$ into its negative. Let $H_t$ $(0 \leq t \leq 1, H = H_0 = -H_1)$ represent the deformation, and let $X_t$ represent the successive positions of a definite point $X$ of $H$. For each value of $t$ the set $K_t = X^{-1} H_t$ is a homeomorph of $H_t$ and hence an oriented cell containing $E$. Since $X_1 = X_0$, it follows that $K_0 = -K_1$. It is clear that the deformation $H_t$ can be modified if necessary so that $K_t$ is always within a given $n$-cell $L$, neighborhood of $E$. Thus $K_t$ defines an isotopic deformation within $L$ of an oriented $n$-cell into its negative, which is impossible.

[1] O. Schreier, *Hamb. Abh.*, **4**, 15–32 (1926).

# ON A POLAR REPRESENTATION OF NON-SINGULAR SQUARE MATRICES

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Let $\mathbf{A}$ be a square matrix of $n$ rows and let the element in the $p^{\text{th}}$ row and $q^{\text{th}}$ column of $\mathbf{A}$ be denoted by $a_{pq}$; defining the square matrix $\mathbf{A}^*$ by the equations $a^*_{pq} = \bar{a}_{qp}$ (where the superposed bar indicates *conjugate complex*) we may construct the two *norms* $\mathbf{N}_1 = \mathbf{AA}^*$ and $\mathbf{N}_2 = \mathbf{A}^*\mathbf{A}$ of $\mathbf{A}$. These norms are in general different but when they are equal the matrix $\mathbf{A}$ is said to be normal. We shall consider only the first norm $\mathbf{N}_1$ in what follows it being clear that our remarks apply equally to $\mathbf{N}_2$.

It is evident that $\mathbf{N}_1 = \mathbf{N}_1^*$, i.e., that $\mathbf{N}_1$ is Hermitian. With any Hermitian matrix $\mathbf{H}$ may be associated a *composite form* $\phi(\mathbf{H}) \equiv \sum_{pq} h_{pq} x_p \bar{x}_q$ assuming real values. The form $\phi(\mathbf{N}_1) = \sum_s (\sum_p a_{ps} x_p)(\sum_p \bar{a}_{ps} \bar{x}_p)$ cannot take negative values and takes the value zero only when all the sums $\sum_p a_{ps} x_p$ are zero. If, as we shall suppose, $\mathbf{A}$ is non-singular this means that $\phi(\mathbf{N}_1)$ takes the value zero only when all the $x$'s are zero, assuming positive values otherwise. In other words $\mathbf{N}_1$ is positively definite. If new variables $y$ are introduced by the formulae $x_p = \sum_s u_{sp} y_s$ where $\mathbf{U}$ is any unitary matrix ($\mathbf{UU}^* = \mathbf{E}$, the unit matrix) $\phi(\mathbf{N}_1)$ is transformed into a composite form whose associated Hermitian matrix is $\mathbf{UN}_1\mathbf{U}^*$ and accordingly this matrix is positively definite ($\mathbf{U}$ being necessarily non-singular). Now the Hermitian matrix $\mathbf{N}_1$ may be transformed[1] by a unitary matrix into a diagonal matrix $[\mathbf{N}_1] = \mathbf{UN}_1\mathbf{U}^*$ and since $[\mathbf{N}_1]$ is positively definite its diagonal elements $\lambda_p$ (which are the characteristic numbers of $\mathbf{N}_1$) are positive. Hence the diagonal matrix $[\mathbf{P}_1]$ whose diagonal elements are $+\sqrt{\lambda_p}$ is positively definite and has its square $= [\mathbf{N}_1]$. This implies that the matrix $\mathbf{P}_1 = \mathbf{U}^*[\mathbf{P}_1]\mathbf{U}$ is positively definite and has its square $= \mathbf{N}_1$. That there exists no other positively definite matrix $\mathbf{Q}$ whose square is $\mathbf{N}_1$ is shown as follows. Any such matrix $\mathbf{Q}$ would have for its characteristic numbers the positive square roots of the characteristic numbers of $\mathbf{N}_1$ and would, accordingly, be unitarily equivalent to $\mathbf{P}_1$ (both matrices having the same characteristic numbers):

$$\mathbf{Q} = \mathbf{VP}_1\mathbf{V}^*; \quad \mathbf{VV}^* = \mathbf{E}.$$

This would imply $\mathbf{N}_1 = \mathbf{VN}_1\mathbf{V}^*$ or, equivalently,

$$[\mathbf{N}_1] = \mathbf{W}[\mathbf{N}_1]\mathbf{W}^*; \quad \mathbf{W} = \mathbf{UVU}^*$$

Since $[\mathbf{N}_1]$ is a diagonal matrix it readily follows by direct computation of $[\mathbf{N}_1]\mathbf{W}$ and $\mathbf{W}[\mathbf{N}_1]$ that if all the characteristic numbers of $\mathbf{N}_1$ are different $\mathbf{W}$ is a diagonal matrix and this implies $\mathbf{Q} = \mathbf{P}_1$. For then $\mathbf{W}[\mathbf{P}_1] = [\mathbf{P}_1]\mathbf{W}$ and so $\mathbf{Q} = \mathbf{V}\mathbf{P}_1\mathbf{V}^* = \mathbf{U}^*\mathbf{W}\mathbf{U}\mathbf{P}_1\mathbf{U}^*\mathbf{W}^*\mathbf{U} = \mathbf{U}^*\mathbf{W}[\mathbf{P}_1]\mathbf{W}^*\mathbf{U} = \mathbf{U}^*[\mathbf{P}_1]\mathbf{U} = \mathbf{P}_1$. If several of the characteristic numbers of $\mathbf{N}_1$ are equal $\mathbf{W}$ need not be a diagonal matrix but will nevertheless be such that $\mathbf{W}[\mathbf{P}_1]\mathbf{W}^* = [\mathbf{P}_1]$ since to each equality among the characteristic numbers of $\mathbf{N}_1$ there is a corresponding equality among the characteristic numbers of $\mathbf{P}_1$.[2] The fact that $\mathbf{Q} = \mathbf{P}_1$ follows by the argument just given. The positively definite matrix thus determined uniquely has a reciprocal $\mathbf{P}_1^{-1}$ and it is clear that the matrix $\mathbf{U} = \mathbf{P}_1^{-1}\mathbf{A}$ is unitary; for $\mathbf{U}\mathbf{U}^* = \mathbf{P}_1^{-1}\mathbf{A}\mathbf{A}^*\mathbf{P}_1^{-1} = \mathbf{E}$. We have, then, the representation $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ of the non-singular matrix $\mathbf{A}$ as the product of a positively definite matrix $\mathbf{P}_1$ by a unitary matrix and the representation is *unique;* for from $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ follows $\mathbf{A}\mathbf{A}^* = \mathbf{P}_1^2$ which (together with the positive definite character of $\mathbf{P}_1$) determines $\mathbf{P}_1$ uniquely. $\mathbf{P}_1$ being determined $\mathbf{U}$ follows unambiguously as $\mathbf{P}_1^{-1}\mathbf{A}$.

Using the second norm we can represent $\mathbf{A}$ unambiguously in the form $\mathbf{V}\mathbf{P}_2$ and it is immediately evident that $\mathbf{V} = \mathbf{U}$; for $\mathbf{A} = \mathbf{V}\mathbf{P}_2 = \mathbf{V}\mathbf{P}_2\mathbf{V}^*.\mathbf{V}$ and $\mathbf{V}\mathbf{P}_2\mathbf{V}^*$ being positively definite it follows from the proven uniqueness of the analysis of $\mathbf{A}$ into the form $\mathbf{P}_1\mathbf{U}$ that $\mathbf{V}\mathbf{P}_2\mathbf{V}^* = \mathbf{P}_1$; $\mathbf{V} = \mathbf{U}$. We state then the following theorem:

*Any non-singular square matrix* $\mathbf{A}$ *may be represented uniquely in the forms*

$$\mathbf{A} = \mathbf{P}_1\mathbf{U} = \mathbf{U}\mathbf{P}_2,$$

*where* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are positively definite and* $\mathbf{U}$ *is unitary. Either of these representations we call a polar representation* (for $n = 1$, $\mathbf{A}$ is an ordinary complex number $a = re^{i\theta}$ and $\mathbf{P}_1$ is the modulus $r$ while $\mathbf{U}$ is the turn $e^{i\theta}$).

The characteristic numbers $\lambda$ of the matrix $\mathbf{A}$ are invariants and *a fortiori* unitary invariants. There are, however, other unitary invariants. The problem of determining the complete system of unitary invariants of an arbitrary square matrix seems to be as yet unsolved[3] but we may state the following theorem:

The characteristic numbers of the "polar coördinates" $\mathbf{P}_1$ and $\mathbf{U}$ of $\mathbf{A}$ are unitary invariants of $\mathbf{A}$.

In fact, from $\mathbf{A} = \mathbf{P}_1\mathbf{U}$ follows $\mathbf{V}\mathbf{A}\mathbf{V}^* = \mathbf{V}\mathbf{P}_1\mathbf{V}^*.\mathbf{V}\mathbf{U}\mathbf{V}^*$ ($\mathbf{V}$ being a unitary matrix) and $\mathbf{V}\mathbf{P}\mathbf{V}^*$ being positively definite with $\mathbf{P}_1$ it follows that $\mathbf{V}\mathbf{P}_1\mathbf{V}^*$ and $\mathbf{V}\mathbf{U}\mathbf{V}^*$ are the polar coördinates of $\mathbf{V}\mathbf{A}\mathbf{V}^*$. Hence any unitary invariant of either $\mathbf{P}_1$ or $\mathbf{U}$ is a unitary invariant of $\mathbf{A}$; in fact if $\phi(\mathbf{P}_1) = F(\mathbf{A})$ is a unitary invariant of $\mathbf{A}$ we have $F(\mathbf{V}\mathbf{A}\mathbf{V}^*) = \phi(\mathbf{V}\mathbf{P}_1\mathbf{V}^*) = \phi(\mathbf{P}_1) = F(\mathbf{A})$ and similarly for $\mathbf{U}$. In particular the symmetric functions of the characteristic numbers of $\mathbf{P}_1$ and $\mathbf{U}$ are unitary invariants of $\mathbf{A}$.

Included among these is the sum of the squares of the characteristic numbers of $\mathbf{P}_1$, i.e., the sum of the characteristic numbers of $\mathbf{N}_1 = \mathbf{AA}^*$; this is the well-known unitary invariant $\sum_{p,q} a_{pq}\bar{a}_{pq}$ of Frobenius.

When $\mathbf{A}$ is normal $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$ or $\mathbf{P}_1\mathbf{U}.\mathbf{U}^*\mathbf{P}_1 = \mathbf{U}^*{}_1\mathbf{P}.\mathbf{P}_1\mathbf{U}$ so that $\mathbf{P}_1^2 = \mathbf{U}^*\mathbf{P}_1^2\mathbf{U} = (\mathbf{U}^*\mathbf{P}_1\mathbf{U})^2$. Hence $\mathbf{P}_1 = \mathbf{U}^*\mathbf{P}_1\mathbf{U}$ or $\mathbf{UP}_1 = \mathbf{P}_1\mathbf{U}$. Conversely if $\mathbf{UP}_1 = \mathbf{P}_1\mathbf{U}$ we have $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{A}$ so that a *matrix* $\mathbf{A}$ *is normal when and only when its polar coördinates are commutable, that is, when the polar representations* $\mathbf{A} = \mathbf{P}_1\mathbf{U}$, $\mathbf{A} = \mathbf{UP}_2$ *coincide.*

It may be mentioned that the above considerations are valid also in the real domain. In this case the polar representation is simply the algebraic formulation of the fact, well known for $n = 3$ from the kinematics of homogeneous linear (non-singular) deformations, that any such deformation may be represented as a superposition of a dilatation and a rotation (the norm $\mathbf{AA}^*$ of $\mathbf{A}$ determining the ellipsoid of dilatation belonging to the deformation $\mathbf{A}$).

[1] See, for example, Weyl, H., *Gruppentheorie und Quantenmechanik*, Leipzig, pp. 19–23, 1928.

[2] See Weyl, H., loc. cit.

[3] Since writing the above this problem has been solved and will be treated in a forthcoming note in these PROCEEDINGS.

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## *NOTE ON THE HEAVISIDE EXPANSION FORMULA*

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The Expansion Formula solution of the linear differential equation with constant coefficients

$$a_0 \frac{d^n x}{dyn} + a_1 \frac{d^{n-1}x}{dy^{n-1}} + \ldots + a_n y = F$$

was first stated by Oliver Heaviside. Perhaps due to his rather obscure methods of presentation, various writers have stated that the formula was given without proof. Nevertheless, Heaviside gave two proofs of the formula which may be traced through his writings. One of these, which we may designate as the second of Heaviside's proofs, was discussed a few years ago by Vallarta.[1] Heaviside really made no clear point of demarcation between his proofs and in all probability did not believe any proof was necessary. The Expansion Formula was but a single result of his devious analyses in the solution of certain differential

equations frequently met in electrical theory. Both proofs hinge on the so-called Conjugate Property of which Heaviside, Rayleigh and Routh made extensive use. Heaviside employed this theorem to great advantage and it serves as a connecting link between the two proofs. The following paragraphs are concerned with the first and restricted proof of the Expansion Formula which up to the present time has received scant attention; later, it will be shown that a proof may be easily and directly established from Rayleigh's[2] investigations of free and forced vibrations.

As has been stated, Heaviside's proof of his Expansion Formula solution is based on the Conjugate Property, which states that the mutual potential and the mutual kinetic energies of two normal systems, $r$ and $s$ are equal at every instant when all the conditions affecting the system are accounted for. Thus symbolically

$$U_{rs} - T_{rs} = 0, \tag{1}$$

where $U_{rs}$ is the mutual potential energy and $T_{rs}$ is the mutual-kinetic energy. Heaviside repeatedly stated the application of this important principle to the solution of linear differential equations. That a proof, constructed along such considerations, should have been so long in forthcoming is indeed rather remarkable. In deriving the Conjugate Property, we shall follow Heaviside's procedure by considering the equations for a line with distributed capacity, inductance and resistance, $C$, $L$, and $r$, respectively, per unit length of line. These may be shown to be

$$\left.\begin{aligned} ri + L\frac{di}{dt} = -\frac{\partial e}{\partial x} = \frac{1}{C}\frac{\partial^2 i}{\partial x^2} \\ C\frac{de}{dt} = -\frac{\partial i}{\partial x} \end{aligned}\right\}. \tag{2}$$

From these an equation containing only $e$ may be obtained, but the two above will suffice. The most general solutions are, by the theory of differential equations,

$$e = \sum Au\epsilon^{pt},\ i = \sum Aw\epsilon^{pt} \tag{3}$$

where $A$ is the subsidence constant, of which we shall hear more later, $u$ and $w$ are the normal functions of potential and current (functions of $x$ only), and $p$ is the operator $\frac{d}{dt}$, a root of the determinental equation. The normal solutions which determine the $p$'s are gotten by putting

$$e = u\epsilon^{pt},\ i = w\epsilon^{pt} \tag{4}$$

in the first of equations (2) above. The resulting equations will determine the $p$'s and is called the determinental equation.

Thus far, the procedure has been in accordance with the steps usually employed in solving certain forms of differential equations. Heaviside's next step was to obtain an expression for the $A$'s in (3) in terms of the initial terminal conditions, and it was at this point that he threw aside the orthodox methods for solving them and arrived at a general expression by means of the Conjugate Property. This theorem and its applicableness to the solution of the $A$'s in (3) was originally presented in a paper[3] published in 1881, but its complete derivation was first given in an article[4] prepared the following year but published only in his Electrical Papers. Returning to equations (3), we can prove by substitution of equations (4) and algebraical composition that

$$\int_{line}(Cu_1u_2 - Lw_1w_2)dx = \frac{\left[w_1w_2\left(\frac{u_1}{w_1} - \frac{u_2}{w_2}\right)\right]_{line}}{p_1 - p_2} \qquad (5)$$

where $u_1$, $u_2$ and $w_1$, $w_2$ are two admissible values belonging to a pair of normal systems consistent with the $p$'s by putting $u$, $w$ and $p$ in (4) first equal to $u_1$, $w_1$, $p_1$ and then equal to $u_2$, $w_2$, $p_2$. This is permissible so long as the $u$'s and $w$'s are consistent with the $p$'s. Now, in (5), let $p_1$ approach $p_2$; then on going to the limit, $u_1 = u_2$ and $w_1 = w_2$ giving

$$\int_{line}(Cu^2 - Lw^2)dx = \left[w^2\frac{d}{dp}Z\right]_{line} \qquad (6)$$

where $Z$ is the function $\frac{u}{w}$. The interpretations are as follows: Equation (5) gives the excess of mutual potential energy over the mutual kinetic energy of two normal arrangements of potential and current so far as the line is concerned. When account is taken of the terminal conditions, the right-hand members of both (5) and (6) vanish. Equation (6) is a special case of (5) when the normal systems are one and the same. The excess of mutual potential over the mutual kinetic energy is thus given in terms of the potential and current at the ends of the line.

If we concern ourselves with the initial conditions, we have from (3), when $t = 0$

$$\left.\begin{aligned} U &= \sum Au \\ W &= \sum Aw \end{aligned}\right\} \qquad (7)$$

$U$ and $W$ are the initial potential and current at one end of the line. Multiplying the first by $Cu$ and the second by $Lw$, these expressions can be shown to follow

$$\int_{line}(CUu - LWw)dx = A\int_{line}(Cu^2 - Lw^2)dx \qquad (8)$$

Thus, utilizing (6)

$$\frac{\int_{line}(CUu - LWw)dx}{\int_{line}(Cu^2 - Lw^2)dx} = \frac{U_{01} - T_{01}}{U_{11} - T_{11}} = \frac{U_{01} - T_{01}}{2(U_1 - T_1)} = \frac{U_{01} - T_{01}}{w^2 \frac{d}{dp} Z} = A \quad (9)$$

where $U_{01}$ and $T_{01}$ are the initial mutual potential and kinetic energies of the initial and normal state as designated by the subscript $_{01}$. By analogy the other terms may be inferred from their proper subscripts. If any energy resides initially in the terminal arrangements, additional terms must be included. As a rule, in most cases met in practice, no energy resides initially at the terminals. Combining (8) and (3), we get

$$i = \sum w \frac{U_{01} - T_{01}}{w^2 \frac{d}{dp} Z} \epsilon^{pt} \quad (10)$$

thus being rid of the $A$'s, which was Heaviside's chief purpose in developing the Conjugate Property. The subsidence solution has now been derived in terms of the normal functions and known conditions. The extension to the derivation of the Expansion Formula was then an easy process. We shall here, however, only briefly state Heaviside's procedure. The reader is referred to either the original discussion[5] employed by Heaviside, or to Vallarta's paper for an excellent treatment.

To obtain the Expansion Formula, having proceeded as far as (10), Heaviside employed an artifice.[5] Taking a circuit with the customary constants, he converts the external forces applied to it into a condenser of infinite capacity, thus creating a subsidence condition. The effect of this substitution is the same as that of a constant applied voltage. By utilizing the conjugate relationship of this system, and substituting in (10) he obtains the final equation of subsidence

$$i = \sum_{j=1}^{j=n} \frac{E\epsilon^{p_j t}}{p_j \frac{d}{dp} Z(p)_{p=p_j}} \quad (11)$$

$$j = 1, 2, 3, \ldots, n.$$

where $E$ is the voltage applied. The complete explicit solution follows thus

$$i = \frac{E}{Z(p)_{p=0}} \sum_{j=1}^{j=n} \frac{E\epsilon^{p_j t}}{p_j \frac{d}{dp} Z(p)_{p=p_j}} \quad (12)$$

$$j = 1, 2, 3, \ldots, n.$$

The first term on the right is the steady state solution. Heaviside at once noted the usefulness of this equation, although the basis for its

deduction was by no means general. We shall now turn to a general proof of the formula based on an analysis due to Raleigh[2] with which Heaviside was undoubtedly familiar. The proof is a digression of the alternative Heaviside procedure which has been discussed by Vallarta.[1]

Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be a set of forces impressed on a system of $n$-degrees of freedom whose coördinates we may designate as $\psi_1, \psi_2, \ldots, \psi_n$. Write the expressions for the potential, dissipative and kinetic energies in terms of these coördinates, thus

$$\left.\begin{aligned} T &= \tfrac{1}{2} A_{11}\dot{\psi}_1^2 + \tfrac{1}{2} A_{22}\dot{\psi}_2^2 + \ldots + A_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ Q &= \tfrac{1}{2} B_{11}\dot{\psi}_1^2 + \tfrac{1}{2} B_{22}\dot{\psi}_2^2 + \ldots + B_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ U &= \tfrac{1}{2} C_{11}\psi_1^2 + \tfrac{1}{2} C_{22}\psi_2^2 + \ldots + C_{12}\psi_1\psi_2 + \ldots \end{aligned}\right\} \qquad (13)$$

$$\dot{\psi} = \frac{\partial \psi}{\partial t}$$

$T$, $Q$ and $U$ are the kinetic energy, dissipative energy and potential energy, respectively, and $A$, $B$ and $C$ are constants. Substitute these in the Lagrange-Rayleigh equation

$$\frac{d}{dt}\left(\frac{\partial^2 T}{\partial \dot{\psi}^2}\right) + \frac{\partial Q}{\partial \dot{\psi}} + \frac{\partial U}{\partial \psi} = \Psi. \qquad (14)$$

We then obtain the following set of equations

$$\left.\begin{aligned} a_{11}\psi_1 + a_{12}\psi_2 + \ldots &= \Psi_1 \\ \ldots\ldots\ldots & \\ a_{n1}\psi_1 + a_{n2}\psi_2 + \ldots &= \Psi_n \end{aligned}\right\} \qquad (15)$$

where, for compactness, the general constant $a_{rs}$ is the operator

$$A_{rs}\frac{\partial^2}{\partial t^2} + B_{rs}\frac{\partial}{\partial t} + C_{rs}.$$

The solution of any one of the variables is then possible by eliminating all the others, provided the constants and the impressed forces are known. In general

$$\left.\begin{aligned} \nabla\psi_1 &= \frac{\partial \nabla}{\partial a_{11}}\Psi_1 + \frac{\partial \nabla}{\partial a_{12}}\Psi_2 + \ldots \\ \ldots\ldots\ldots & \\ \nabla\psi_n &= \frac{\partial a}{\psi \nabla_{n1}}\psi, + \frac{\psi \nabla}{\partial a_{n2}}\Psi_2 + \ldots \end{aligned}\right\} \qquad (16)$$

where $\Delta$ defines the determinant

$$\nabla \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots \end{vmatrix}. \tag{17}$$

Without any loss of generality, we may put all the forces but one equal to zero. Then we have

$$\nabla\psi_1 = \frac{\partial\nabla}{\partial a_{11}}\Psi_1, \tag{18}$$

or, to avoid subscripts, and putting $Z$ for $a_{11}$

$$\nabla\psi = \frac{d\nabla}{dZ}\Psi. \tag{19}$$

Now, if all the impressed forces were zero, the determinant (17) above would vanish. That is $\nabla = 0$. If in equations (15), we attempt to obtain the normal solutions by putting $\psi = we^{pt}$ after the impressed forces are put equal to zero in them, we obtain the determinental equations of the $p$'s; thus

$$\nabla(p) = 0. \tag{20}$$

Further, suppose that

$$\Psi = Ee^{pt} \tag{21}$$

which is permissible and substitute in (19) after the $p$'s have been written into $\nabla$

$$\psi = \frac{1}{\nabla(p)}\frac{d\nabla(p)}{dZ}Ee^{pt}. \tag{22}$$

If next we assume $\nabla(p)$ is complex, being made up of two components, one real and one imaginary, we may write

$$\nabla(p) = \nabla_1(p) + jp\nabla_2(p), \quad j = \sqrt{-1} \tag{23}$$

$\nabla_1$ and $\nabla_2$ are functions of $\nabla$. The first necessarily vanishes, since the period of the force is identical to that obtained from the determinental equation. Therefore,

$$\left.\begin{aligned} \nabla(p) &= jp\nabla_2(p) \\ \frac{d\nabla(p)}{dZ} &= j\nabla_2(p)\frac{dp}{dZ} + jp\frac{d\nabla_2(p)}{dZ} \end{aligned}\right\}. \tag{24}$$

The last term on the right of the second equation is also zero. Substituting these in (22) we get

$$\psi = \frac{1}{jp\nabla_2(p)}j\nabla_2(p)\frac{dp}{dZ}Ee^{pt} \tag{25'}$$

or if $i$ is written for $\psi$

$$i = \sum \frac{E\epsilon^{pt}}{p\frac{dz}{dp}}. \qquad (25)$$

The $\sum$ denotes summation over the roots of the determinental equation. Equation (25) is identical to the equation of subsidence obtained in (11) according to Heaviside's procedure. It has been deduced, however, without the application of the Conjugate Theorem, although the same limitations hold, namely, that there are no null or repeated roots of $p$.

[1] M. S. Vallarts, "Heaviside's Proof of His Expansion Theorem," *Jl. A. I. E. E*, April, 1926.

[2] Lord Rayleigh, "Scientific Papers," **1**, 176–187; and "Theory of Sound," 103–142.

[3] Oliver Heaviside, "Electrical Papers," **1**, "On Induction between Parallel Wires," 127–129.

[4] *Ibid.*, "Contributions to the Theory of the Propagation of Current in Wires," 142–148.

[5] Loc. cit., 3, **2**, 372–374.

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## *THE DISTRIBUTION OF CHI-SQUARE*

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R. A. Fisher[1] gives a table of $\chi^2$ and states that for large values of $n$, the number of degrees of freedom in the distribution,

$$\sqrt{2\chi^2} - \sqrt{2n-1} \text{ is normally distributed with } \sigma = 1. \qquad (1)$$

It is interesting to ask what other formulas of a similar sort might be used.

When the integrand $f(x)$ of a definite integral vanishes at the limits and has a single maximum, a useful approximation to the value of the integral can sometimes be found by expanding $\log f(x)$ about its maximum $x = m$, writing

$$\varphi(x) = \log f(x) = \varphi(m) + \phi'(x-m) + {}^1/_2\phi''(m)(x-m)^2 + \ldots$$

$$\int_a^b f(x)dx = \int_a^b e^{\varphi(m)} e^{\frac{1}{2}\varphi''(m)(x-m)^2} dx \text{ (approx.)}$$

$$\text{or } \int_a^b f(x)dx = e^{\varphi(m)} \frac{\sqrt{2\pi}}{\sqrt{-\varphi''(m)}}$$

$$\text{or } \log \int_b^b f(x)dx = \varphi(m) + \frac{1}{2}\log 2\pi - \frac{1}{2}\log\left[-\varphi''(m)\right]$$

The assumption is that the higher terms in the expansion of $\varphi(x)$ contribute small amounts to the integral and that the limits $a$ and $b$ are sufficiently remote from the maximum $x = m$ so that the integral may be regarded as a complete probability integral.

The usual application of this method is to the gamma function

$$\Gamma(n) = (n-1)! = \int_0^\infty x^{n-1}e^{-x}dx = \int_0^\infty py^{np-1}e^{-y^p}dy$$

where $x = y^p$. Then

$$\varphi(y) = \log f(y) = \log p + (np-1)\log y - y^p$$
$$m = (n-1/p)^{1/p},\ -\varphi''(m) = p^2(n-1/p)^{1-2/p},$$
$$\varphi(m) = \log p + (n-1/p)\log(n-1/p) - (n-1/p),$$

$$\log\Gamma(n) = \log(n-1)! = \frac{1}{2}\log 2\pi + \left(n-\frac{1}{2}\right)\log\left(n-\frac{1}{p}\right) - \left(n-\frac{1}{p}\right). \quad (2)$$

When $p = 1$ this expression gives the well-known Stirling's formula; when $p = 2$ it gives a somewhat better approximation.[2] One may observe that $\Gamma(n)$ is defined from $n = 0+$ to $n = \infty$, whereas Stirling's formula is defined only from $n = 1+$ on and the alternative formula only from $n = (1/2)+$ on. The method gives an approximation to $\Gamma(n)$ over the whole range of the function if $p = \infty$ and

$$\log\Gamma(n) = \log(n-1)! = \frac{1}{2}\log 2\pi + \left(n-\frac{1}{2}\right)\log n - n$$

or

$$\Gamma(n+1) = n! = \sqrt{\frac{2\pi}{n+1}}\left(\frac{n+1}{e}\right)^{n+1}.$$

The following table shows the values of $\log_{10}\Gamma(n)$ as computed for some small values of $n$ with $p = 1, 2, 5, \infty$.

| | $n = 1/2$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|---|---|---|---|---|---|---|
| $p = 1$ | imag | $\infty$ | −0.03520 | 0.28076 | 0.76613 | 1.37118 |
| $p = 2$ | $\infty$ | 0.03143 | 0.01178 | 0.30813 | 0.78330 | 1.38420 |
| $p = 5$ | 0.26080 | 0.00320 | 0.00026 | 0.30097 | 0.77800 | 1.38006 |
| $p = \infty$ | 0.18194 | −0.03520 | −0.01796 | 0.28901 | 0.76912 | 1.37298 |
| $\log_{10}\Gamma(n)$ | 0.24857 | 0.00000 | 0.00000 | 0.30103 | 0.77815 | 1.38021 |

It is seen that for small values of $n$ the approximation given by $p = \infty$ is decidedly better than that figured from Stirling's formula ($p = 1$) though not so good as that for $p = 2$ which corresponds to the other formula in common use. It is especially interesting to see that with $p = 5$ the approximation is out of all proportion better than any of the others.

The distribution of $\chi^2$ is governed by the equation for the differential frequency as

$$dF = C\chi^{n-1}e^{-\chi^2/2}d\chi \quad (3)$$

where $C$ is so adjusted that the integral of $dF$ from 0 to $\infty$ is 1. The possibility of obtaining so good an approximation to the complete integral, which is $\Gamma(n/2)$ except for a multiplier, suggests that the distribution might be obtained in a similar way.

$$dF = 2^{n/2-1}C\left(\frac{1}{2}\chi\right)^{2(n/2-1)} e^{-(\chi^2/2)}\, d\left(\frac{1}{2}\chi^2\right)$$

$$= 2^{n/2-1}Cpy^{np/2-1}\, e^{-y^p}\, dy \qquad (3')$$

where $$y^p = (\chi^2/2).$$

The maximum is at

$$y = m = \left(\frac{n}{2} - \frac{1}{p}\right)^{1/p} \text{ with } \sigma = \frac{1}{p(n/2-1/p)^{1/2-1/p}}.$$

The question is whether $y = (\chi^2/2)^{1/p}$ may be regarded as a normal variate distributed about the mean $m$ with the indicated value of $\sigma$. To obtain the formula analogous to Fisher's one should put $p = 2$ and multiply the mean and standard deviation by 2. We should have

$$\sqrt{2\chi^2} \text{ about mean } \sqrt{2n-2} \text{ with } \sigma = 1.$$

A calculation of a comparative table shows that this result is not so good as his. If one takes $p = 3$ the analysis suggests the normal distribution of

$$\sqrt[3]{\chi^2/2} \text{ about } \sqrt[3]{n/2-1/3} \text{ with } \sigma = \frac{1}{3\sqrt[6]{n/2-1/3}}$$

or

$$\sqrt[3]{\chi^2} \text{ about } \sqrt[3]{n-2/3} \text{ with } \sigma = \frac{\sqrt{2}}{3\sqrt[6]{n-2/3}}. \qquad (4)$$

The result is better than that given by Fisher's formula in some parts of the table and worse in others as shown in table 1. A considerable trial with values of $p$ other than 3 indicates no improvement of significance.

There is another method of attack. One may write

$$\chi^2 = n + e, \quad \chi^{2/p} = (n+e)^{1/p}$$

The expansion by the binomial theorem goes according to powers of $e$ or effectively of $e/n$, the mean value of $e$ is zero, the mean of $e^2$ is the second moment of $\chi^2$ about its mean value $n$ and equals $2n$, the mean of $e^k$ is the $k$th moment of $\chi^2$ about its mean and these moments may all be obtained in terms of the $\Gamma$-functions by integration of (3) multiplied by $\chi^{2k}$. The algebra is fairly long but straightforward and the final results are rational in $n$. Hence the mean value of $\chi^{2/p}$ may be obtained as

$$\text{mean } \left(\frac{\chi^2}{n}\right)^{\frac{1}{p}} = 1 + \frac{1}{n}\frac{1}{p}\left(\frac{1}{p}-1\right) + \frac{4}{3}\frac{1}{n^2}\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right) + \ldots.$$

From this expression and the original expansion the moments of $(\chi^2/n)^{1/p}$ about its mean can be found as

$$\text{2nd moment} = \frac{2}{np^2} + \frac{2}{n^2p^2}\left(\frac{1}{p} - 1\right)\left(\frac{3}{p} - 1\right) + \ldots$$

$$\text{3rd moment} = \frac{4}{n^2}\left(\frac{3}{p} - 1\right) + \ldots$$

From these results it appears that if $p = 3$ the 3rd moment vanishes to the order $1/n^3$, and the term of order $1/n^2$ in the second moment also vanishes. This suggests that to a considerable degree of approximation, increasing as $n$ increases, we may assume that

$$\left(\frac{\chi^2}{n}\right)^{1/3} \text{ is normally distributed about } 1 - \frac{2}{9n} \text{ with } \sigma^2 = \frac{2}{9n}. \qquad (5)$$

It is interesting to compare this result with the tabular values of $\chi^2$ and with the results of using (1) given by Fisher or (4) which was obtained by a different method. The comparison is given in table 1.

TABLE 1

VALUES OF $\chi^2$ FOR $n$ = 1, 2, 3, 10, 30, AT $P$ = 0.80, 0.50, 0.20, 0.05, 0.01
True values marked $T$ with those given by formulas (5), (1), (4)

| $n$ | | $P = 0.80$ | $P = 0.50$ | $P = 0.20$ | $P = 0.05$ | $P = 0.01$ |
|---|---|---|---|---|---|---|
| | $T$ | 0.0642 | 0.455 | 1.642 | 3.841 | 6.635 |
| 1 | (5) | 0.0553 | 0.470 | 1.618 | 3.747 | 6.586 |
| | (1) | 0.0125 | 0.500 | 1.696 | 3.498 | 5.532 |
| | (4) | 0.0102 | 0.333 | 1.600 | 4.287 | 8.119 |
| | $T$ | 0.446 | 1.386 | 3.219 | 5.991 | 9.210 |
| 2 | (5) | 0.450 | 1.405 | 3.195 | 5.936 | 9.220 |
| | (1) | 0.396 | 1.500 | 3.312 | 5.702 | 8.235 |
| | (4) | 0.378 | 1.333 | 3.232 | 6.222 | 9.869 |
| | $T$ | 1.005 | 2.366 | 4.642 | 7.815 | 11.341 |
| 3 | (5) | 1.015 | 2.381 | 4.622 | 7.775 | 11.370 |
| | (1) | 0.972 | 2.500 | 4.736 | 7.531 | 10.171 |
| | (4) | 0.946 | 2.333 | 4.664 | 7.995 | 11.826 |
| | $T$ | 6.179 | 9.342 | 13.442 | 18.307 | 23.209 |
| 10 | (5) | 6.191 | 9.349 | 13.419 | 18.298 | 23.246 |
| | (1) | 6.186 | 9.500 | 13.523 | 18.023 | 22.346 |
| | (4) | 6.155 | 9.333 | 13.451 | 18.372 | 23.381 |
| | $T$ | 23.364 | 29.336 | 36.250 | 43.773 | 50.892 |
| 30 | (5) | 23.376 | 29.340 | 36.237 | 43.770 | 50.913 |
| | (1) | 23.389 | 29.500 | 36.318 | 43.487 | 50.074 |
| | (4) | 20.984 | 29.333 | 36.258 | 43.815 | 50.986 |

Table 1 shows that the distribution (4) obtained from the differential (3′) appears to be better than (1) in some places and worse in others; as it is not so simple it should be rejected. On the other hand the distribution (5) obtained from expansion is decidedly better than (1) in most parts of the tables, and is indeed so good as to make it nearly equivalent to the tabulated values of $\chi^2$ for all values of $n$ from $P = 0.80$ to $P = 0.01$.

Thus a probability integral table taken with the rule (5) can for most purposes replace a table of $\chi^2$. Indeed if the aim is merely to test for "significance" with $P = 0.05$ the point of distinction between non-significance and significance as is customary, we do not need a probability integral table but merely the rule that for significance

$$\left(\frac{\chi^2}{n}\right)^{1/3} - \left(1 - \frac{2}{9n}\right) \text{ shall exceed } 1.65\sqrt{\frac{2}{9n}}.$$

It is somewhat remarkable that (5) gives so good a representation as it does over so wide a range, because the distribution of $\chi^{2/3}$ is really not normal as may be seen from tabulation below, which gives the true mean,[3] the approximate mean, the true value of $\sigma^2$ and the approximate value, and the true values of $\beta = \mu_3/\sigma^3$ which has been taken as 0 and of $\beta_2 = \mu_4/\sigma^4$ which has been taken as 3 in the approximation. For small values of $n$ the Charlier A-type expansion would have considerable terms in addition to the first.[4]

| $n$ | Mean | $1 - \frac{2}{9n}$ | $\sigma^2$ | $\frac{2}{9n}$ | $\beta$ | $\beta_2$ |
|---|---|---|---|---|---|---|
| 1 | 0.80238 | 0.77778 | 0.18704 | 0.22222 | +0.417 | 2.68 |
| 2 | 0.89298 | 0.88889 | 0.10533 | 0.11111 | +0.168 | 2.73 |
| 3 | 0.92723 | 0.92593 | 0.07226 | 0.07407 | +0.093 | 2.80 |
| 10 | 0.97782 | 0.97778 | 0.02217 | 0.02222 | +0.012 | 2.97 |
| 30 | 0.99259 | 0.99259 | 0.007407 | 0.007407 | −0.004 | 3.14 |

[1] R. A. Fisher, *Statistical Methods for Research Workers*, 2nd Edition, 1928, pp. 96–7.

[2] See E. B. Wilson, *Advanced Calculus*, p. 384 and p. 386, Ex. 18.

[3] The true moments of $\chi^2$ may obviously be expressed in terms of the $\Gamma$-function.

[4] In many applications of the $\chi^2$-test the number of entries in some cell may be fairly small and in that case the test may not work very well. See "Goodness of Fit," by Wilson, Hilferty and Maher, *J. Amer. Statistical Assoc.*, December, 1931. From a practical point of view this has to be borne in mind in judging how accurate an approximation to the distribution of $\chi^2$ is necessary for practical purposes.

# RELATIONS BETWEEN CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Some of the functions mentioned in a recent paper may be expressed in terms of known functions.

The function $H_{2n}(x)$, which was required to be such that

$$\int_0^\infty e^{-\lambda x} H_{2n}(x)\, dx = \left(\frac{\lambda+1}{\lambda-1}\right)^n \frac{1}{\lambda^2-1},$$

is readily seen to be given by the equation

$$H_{2n}(x) = xe^{-x} F(n+1;\ 2;\ 2x) \tag{1}$$

where $F(\alpha;\ \gamma;\ z)$ denotes Kummer's function

$$1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots$$

which Kummer himself denotes by the symbol $\phi(\alpha, \gamma, z)$. Kummer's function $K(a, b, z)$ may be expressed in terms of Whittaker's function $W_{k,m}(z)$ by comparing his expression for $K$ in terms of $\phi$ with the relation at the end of §16.41 of Whittaker and Watson's "Modern Analysis." When $z > 0$ the relation is

$$2^{2m}\,\Gamma(1/2 + m + k)K(2m, -2k, z) = \pi z^{m-1/2} W_{k,m}(z) \tag{2}$$

This relation, combined with the relation just mentioned, should take the place of equation (32) of my recent paper, which is marred by some slips. In particular,

$$k_{2n}(z)\ \Gamma(1+n) = W_{n,1/2}(2z) \tag{3}$$

and the asymptotic expansion of the function $k$ is consequently (for $x > 0$)

$$k_{2n}(x) \sim \frac{1}{\Gamma(1+n)} e^{-x}(2x)^n \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{n(n-1)^2 \ldots (n-m+1)^2(n-m)}{m!\,(2x)^m} \right\} \tag{4}$$

When $n$ is a positive integer the series ends and we have

$$k_{2n}(x) = (-1)^{n-1}\, 2x.e^{-x} F(1-n;\ 2;\ 2x) \tag{5}$$

From the definition of the function $h$ it is readily seen that when $a > 0$

$$\int_0^\infty e^{-ax} h_{-2n}(x)dx = \int_0^\infty e^{-ax} dx \int_0^\infty e^{-x \coth z - 2nz} dz$$

$$= \int_0^\infty \frac{e^{-2nz} dz}{a + \coth z}$$

$$= \frac{1}{2n} \cdot \frac{1}{a+1} - \frac{1}{n+1} \frac{1}{(a+1)^2} - \frac{1}{n+2} \frac{a-1}{(a+1)^3} - \ldots \quad (6)$$

Now the last series is known to be equal to the integral

$$\int_0^\infty e^{-ax} k_{2n}(x)\, dx \,.\, \frac{\pi}{2} \operatorname{cosec}(n\pi)$$

and so it is readily seen that when $x > 0$, $n > 0$,

$$k_{-2n}(x) = \frac{2}{\pi} \sin(n\pi)\, h_{2n}(x). \quad (7)$$

This relation is equivalent to equation (36) of my former paper in which $k_{-2n}(x)$ should appear instead of $k_{2n}(x)$. Whether $n$ is a positive integer or not we may write

$$h_{2n}(x) = \frac{1}{2} \Gamma(n) W_{-n,\frac{1}{2}}(2x) \qquad \left.\begin{matrix} n > 0 \\ x > 0 \end{matrix}\right\} \quad (8)$$

and the asymptotic expansion for the function $h$ is accordingly

$$h_{2n}(x) \sim \frac{\Gamma(n) e^{-x}}{2^{n+1} x^n} \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{n(n+1)^2 \ldots (n+m-1)^2 (n+m)}{m!\ (2x)^m} \right\}. \quad (9)$$

This expression fails when $n = 0$ for then

$$h_0(x) = \frac{1}{2} [\Gamma'(1) - \log 2]\, e^{-x} - e^x \int_x^\infty e^{-2\xi} \log \xi\, d\xi \quad (10)$$

and the asymptotic expansion is

$$h_0(x) \sim \frac{1}{2} e^{-x} \left[ \Gamma'(1) - \log(2x) - \frac{1}{(2x)} + \frac{1!}{(2x)^2} - \frac{2!}{(2x)^3} + \ldots \right]. \quad (11)$$

# GROUPS INVOLVING A SMALL NUMBER OF CONJUGATES

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If a finite group $G$ has the property that each of its operators besides the identity is transformed into itself only by its own powers then its order $g$ must be either a prime number or the product of distinct prime numbers since every Sylow group involves invariant operators besides the identity. Hence $G$ must involve an invariant subgroup of order $p$, where $p$ is the largest prime number which divides $g$. If $q$ is the largest prime number besides $p$ which divides $g$ than $G$ involves the non-abelian group of order $pq$, and $p-1$ is a multiple of $q$. As a similar remark applies to the quotient group of $G$ with respect to its subgroup of order $p$ it results that this quotient group is of order $q$ whenever the order of $G$ exceeds $p$. This proves the following theorem: *The cyclic group of prime order and the non-cyclic group whose order is the product of two distinct prime numbers are the only groups which have the property that each of the operators of the group besides the identity is transformed into itself under the group by only the powers of this operator.*

It is easy to see that a group $G$ could not involve one and only one complete set of conjugate non-invariant operators since such a set would involve at most $g/2$ of the operators of $G$ as every operator of a group is transformed into itself by its own powers. It could not involve less than $g/2$ of these operators since the central of a non-abelian group is a proper subgroup thereof. The fact that the number of operators in such a set of conjugates could not be exactly $g/2$ is included in a more general theorem relating to all the groups which involve a complete set of conjugate operators composed of exactly one-half of the operators of the group, irrespective of the number of operators which may appear in other sets of conjugates or the number of these sets.

Suppose that $G$ contains a set of conjugates composed of $g/2$ of its operators. As each of these operators is transformed into itself by only two of the operators of $G$ it must be of order 2 whenever $g > 2$. In what follows it will be assumed that this trivial condition is satisfied and hence there is more than one operator of order 2 in the given complete set of conjugates. If $s_1$ and $s_2$ are two such operators their product must have an odd order since each of these two operators is transformed into itself only by its powers. Hence exactly $g/2$ of the operators of $G$, including the identity, are of odd order, and $g$ is not divisible by 4. The operators of odd order in $G$ must therefore constitute a subgroup of index 2 and this subgroup must be abelian since each of its operators corresponds to its

inverse in an automorphism of $G$. This proves the following theorem: *A necessary and sufficient condition that a group contains a complete set of conjugate operators composed of as many as one-half of the operators of the group is that it is either a dihedral group or a generalized dihedral group, whose order is twice an odd number in each case.*[1]

The number of complete sets of conjugate operators, including the identity, in the groups defined by this theorem is obviously $g/4 + 3/2$, but each of these groups contains only one complete set of conjugate non-invariant subgroups, viz., the subgroups of order 2. Hence it results that while there is no group which has the property that it involves one and only one complete set of non-invariant conjugate operators there is at least one infinite system of groups which separately involve one and only one complete set of non-invariant conjugate subgroups. It is possible to determine as follows all the groups which have this property.

Let $G$ be a group which contains one and only one complete set of conjugate non-invariant subgroups and let $H$ be one of the subgroups of this set. It is easy to see that $H$ must be a cyclic prime power subgroup of $G$ and hence its order may be assumed to be $p^m$, $p$ being a prime number. All the conjugates of $H$ must involve a common subgroup $H_0$ of order $p^{m-1}$, which is a characteristic subgroup of $G$. It will first be assumed that $g$ is not a power of $p$. Hence $H$ is a Sylow subgroup of $G$. For if it were not such a subgroup $G$ would be the direct product of its Sylow subgroups since it would involve only one such subgroup of a given order. Hence $G$ would involve more than one set of non-invariant conjugate subgroups which is contrary to the hypothesis. The invariant subgroup generated by $H_0$ and the Sylow subgroups whose orders are prime to $p$ must be the direct product of its Sylow subgroups, abelian and of index $p$ under $G$ since each of it subgroups is invariant under $G$.

Some of the results of the preceding paragraph may be stated in the form of a theorem as follows: *If a group $G$ whose order is divisible by at least two different prime numbers contains one and only one complete set of non-invariant conjugate subgroups then these subgroups must be cyclic Sylow subgroups of $G$ and their cross-cut must be the central of $G$ and of prime index under one of these Sylow subgroups. Moreover, $G$ must involve an abelian subgroup of index $p$ all of whose operators are transformed into the same power by one of the remaining operators of $G$.* On the other hand, if an abelian group which involves a cyclic Sylow subgroup of order $p^{m-1}$ and other operators admits an automorphism of order $p$ in which every operator of this cyclic subgroup corresponds to itself while no other operator has this property, and each of the operators corresponds to the same power of itself, then this abelian group can be extended by an operator of order $p^m$ so as to obtain a group which involves one and only one complete set of non-invariant conjugate subgroups. These subgroups generate $G$ and include all of its operators of order $p^m$.

When $g = p^m$ the complete set of non-invariant conjugate subgroups of $G$ cannot generate $G$ since a prime power group cannot be generated by a single complete set of conjugates under the group. Hence these conjugates generate an invariant subgroup $H_1$. Since in a prime power group every subgroup of a set of conjugates under the group is transformed into itself by at least one of these conjugates besides itself it results that two of these conjugates subgroups must generate $H_1$ since they generate an invariant subgroup of $G$. All the subgroups of $H_1$ are invariant under $H_1$ and $H_1$ involves a cyclic subgroup of index $p$. It is therefore either the quaternion group or the abelian group of order $p^k$ and of type $(k\text{-}1, 1)$, and $G$ involves exactly $p$ conjugates of $H$. It has therefore been proved that if a group of order $p^m$, $p$ being a prime number, involves one and only one complete set of non-invariant conjugate subgroups then it involves a subgroup of index $p$ in which every one of its own subgroups is invariant.

As this subgroup of index $p$ could not be a Hamiltonian group it must be abelian and the commutator subgroup of $G$ must be of order $p$. Hence $G$ contains $p + 1$ abelian subgroups of index $p$ and $H_0$ appears in the central of $G$. Every non-invariant operator of $G$ which is not found in $H_1$ generates the commutator subgroup of $G$. If the quotient group $G/H_1$ were non-cyclic it would be possible to find two non-commutative operators of $G$ such that neither is in $H_1$ and that at least one does not generate the commutator subgroup of $G$. Since this is impossible it has been proved that $G/H_1$ is cyclic and hence $G$ is generated by two operators one of which is a generator of $H$. From this it follows that $H$ is of order $p$, and hence we have established the following theorem: *If a group of order $p^m$, $p$ being a prime number, involves one and only one set of non-invariant conjugate subgroups it must be the non-abelian group of order $p^m$ which involves an operator of order $p^{m-1}$, where $m > 3$ when $p = 2$.*

It was noted above that there is no group which contains one and only one complete set of non-invariant conjugate operators. If a group contains two and only two such sets one of these sets must involve one-half of the operators of $G$ and hence $G$ must be the symmetric group of order 6. In general, the number of operators in a complete set of conjugates must be $g/g_i$, where $g_i$ is the order of the subgroup of $G$ which transforms one of these conjugates into itself. As the order of the central of $G$ cannot exceed $g/4$ when $G$ is non-abelian it results that if $g_1, g_2, \ldots, g_n$ are the orders of the subgroups of $G$ such that one such subgroup corresponds to each complete set of non-invariant conjugate operators then

$$1 > \frac{1}{g_1} + \frac{1}{g_2} + \ldots + \frac{1}{g_n} \geqq \frac{3}{4}$$

When $n$ is a fixed number the values of $g_1, g_2, \ldots, g_n$ must be finite and

the order of the central of $G$ must be finite and hence $g$ is finite. That is, *there is only a finite number of groups which contain a fixed number, greater than zero, of complete sets of non-invariant conjugate operators.* From the theorem at the close of the preceding paragraph it results that there is an infinite number of groups such that each group contains one and only one complete set of non-invariant conjugate subgroups.

When $G$ contains three and only three complete sets of non-invariant conjugate operators it results that one of the three numbers $g_1$, $g_2$, $g_3$ is at most as large as 4. If this smallest number is 2 then $G$ is the dihedral group of order 10. When it is 3 then the central of $G$ must be the identity as this is obviously always true when at least one of the numbers $g_1$, $g_2$, . . ., $g_n$ is a prime number. Hence the order of $G$ is 12 and $G$ is the tetrahedral group. Finally, when the smallest of the three numbers $g_1$, $g_2$, $g_3$ is 4 then each of them is 4 and $G$ is either the octic group or the quaternion group. There are therefore four and only four groups which have the property that each of them contains three and only three complete sets of non-invariant conjugate operators, viz., the dihedral group of order 10, the tetrahedral group, the octic group and the quaternion group.

[1] Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 168.

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## *LINKAGE VALUES IN AN INTERCHANGE COMPLEX IN ZEA**

By Marcus M. Rhoades

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Communicated October 26, 1931

A translocation between non-homologous chromosomes may be either simple or reciprocal. In a simple translocation one chromosome is the donor and a second chromosome the recipient of a piece of the first, but in a reciprocal translocation pieces of chromatin are interchanged between the two chromosomes. Another name for reciprocal translocation is segmental interchange.

The first suggestion of segmental interchange was of a cytological nature and was made by Belling and Blakeslee[1] in their study of chromosome pairing in certain trisomic races in Datura. This hypothesis was later successfully applied by many investigators to explain unusual genetical and cytological behavior in such diverse forms as Stizolobium, Pisum, Oenothera, Zea and Drosophila.

Brink[2] and Brink and Burnham[3] reported a case of semi-sterility in maize. They advanced the hypothesis that the abortion of half the

eggs and pollen was caused by a simple or reciprocal translocation between two non-homologous chromosomes. Brink[4] found that the *P-br* and *b-lg* linkage groups were involved. He reported, however, no linkage between factors in the *P-br* chromosome with factors in the *b-lg* chromosome in cultures segregating for genes in these groups and for semi-sterility. Burnham[5] further substantiated the hypothesis of a segmental interchange by showing that semi-sterile individuals had a ring of four chromosomes at diakinesis. This case of semi-sterility was designated semi-sterile-1. McClintock[6] first supplied a cogent demonstration of a segmental interchange by her determination of the location of the point of breakage between two non-homologous chromosomes in semi-sterile-2.

The purpose of this paper is to report the results obtained in the analysis of another semi-sterile, called semi-sterile-4. Genetical data of a nature not previously reported are presented. The genetical data and cytological behavior may be regarded as conclusive evidence of the occurrence of a segmental interchange.

Genetic tests have shown that the *b-lg* and *pr-v*$_2$ chromosomes are involved. The percentages of crossing-over of genes located in these two linkage groups with the locus of the break or interchange are given in table 1.

TABLE 1

PERCENTAGES OF CROSSING-OVER BETWEEN GENES IN THE *b-lg* AND *pr-v*$_2$ CHROMOSOMES AND THE LOCUS OF THE BREAK

| *b-lg* CHROMOSOME | *pr-v*$_2$ CHROMOSOME |
|---|---|
| *ts*$_1$-break = 4.0 ± 1.20 | *pr*-break = 18.3 ± 1.19 |
| *b*-break = 22.1 ± 0.94 | *v*$_2$-break = 46.3 ± 1.15 |
| *lg*-break = 47.3 ± 1.49 | |

These data show that the locus of the break in the *b-lg* chromosome was very close to *ts*$_1$ which occupies the left-most region in the known genetic map. It is impossible to tell from the present data whether the break in the *b-lg* chromosome took place to the left or right of the *ts*$_1$ locus. In the *pr-v*$_2$ chromosome, if the locus of *v*$_2$ be considered to be to the right of *pr*, then the break took place 18 units to the left of *pr* in what is at present an unmapped portion of the chromosome. The standard crossover values and the values found in the segmental interchange are given in table 2.

TABLE 2

PERCENTAGES OF CROSSING-OVER IN NORMAL LINES AND IN LINES WITH THE SEGMENTAL INTERCHANGE FOR FACTORS IN THE *b-lg* AND *pr-v*$_2$ LINKAGE GROUPS

| NORMAL CROSSOVER VALUES | INTERCHANGE CROSSOVER VALUES |
|---|---|
| *ts*$_1$-*b* = 23 ± % | *ts*$_1$-*b* = 21 ± % |
| *ts*$_1$-*lg* = 40 ± % | *ts*$_1$-*lg* = 40 ± % |
| *b-lg* = 35 ± % | *b-lg* = 34 ± % |
| *pr-v*$_2$ = 42 ± % | *pr-v*$_2$ = 44 ± % |

The data in table 2 show that crossing-over in plants with the interchange is of the same order as in normal lines, since the values found are surprisingly close to the standard distances.

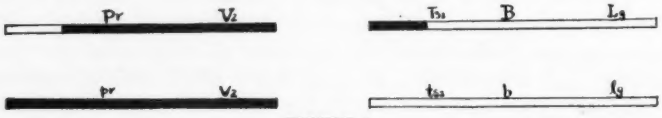


FIGURE 1

Diagrammatical representation of the four chromosomes involved in semi-sterile-4.

Some of the cultures segregating for semi-sterility were also segregating for the factor *b* in the *b-lg* group and the factors *pr* and $v_2$ in the *pr*-$v_2$ linkage group. The data in table 1 show the locus of *b* to be approximately 22 units from the locus of the break in the *b-lg* chromosome and *pr* to be about 18 units from the break in the *pr*-$v_2$ chromosome, so a linkage between *b* and *pr* is expected in those families segregating for the two genes and for semi-sterility. If no double crossovers occur the two factors should be 40 ± units apart. The value obtained was 31 ± percent of crossing-over. The loci of *b* and $v_2$ showed random assortment which is in agreement with the data as $v_2$ was 46 ± crossover units from the break.

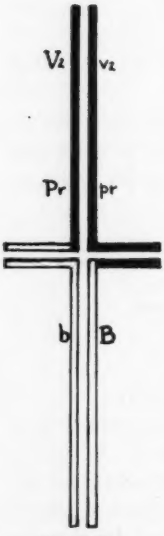


FIGURE 2

Quadripartite figure formed by interchange complex in first meiotic division.

The genetically determined constitution of the four chromosomes forming a ring at diakinesis may be diagrammed as in figure 1.

These four chromosomes in the mid-prophase of the first meiotic division form a quadripartite figure which may be represented as in figure 2, disregarding all loci save those of *pr*, $v_2$ and *b*.

Let the distance from *Pr* to the break be called region 1 and let the distance from *B* to the break be region 2. As functional gametes are formed only when alternate chromosomes in the ring pass to the same pole, the following types of gametes are possible:

| | | |
|---|---|---|
| Non-crossover gametes | *pr b* | normal chromosomes |
| | *Pr B* | interchanged chromosomes |
| Region 1 crossovers | *pr B* | interchanged chromosomes |
| | *Pr b* | normal chromosomes |
| Region 2 crossovers | *pr B* | normal chromosomes |
| | *Pr b* | interchanged chromosomes |

Regions 1 and 2 crossover.....*pr b* interchanged chromosomes
*Pr B* normal chromosomes

Since the plants carrying the interchange complex shown in figure 2 were crossed with *pr b* plants from normal lines, the gametic and zygotic frequencies were the same. If a crossover in region 1 does not interfere with a crossover in region 2 the number of observed double crossovers should be equal to the calculated number with no interference. Actually a coincidence value of 1.1 was found which indicates the absence of interference among two of the four arms of the interchange complex. The detailed data will be published later.

The cytological observations are as follows: Semi-sterile-4 plants show a ring of four chromosomes at diakinesis.† When semi-sterile-4 plants are crossed with races trisomic for the two chromosomes, respectively, the 21 chromosome plants carrying the interchange show cha ns of five chromosomes. Semi-sterile-4 crossed with semi-sterile-1 gives a ring of six chromosomes according to expectations, since the *b-lg* chromosome is common to both cases of segmental interchange. Prophase figures of the interchange complex of semi-sterile-4, first found by Miss Harriet Creighton, show the interchange to be approximately an equal one. The genetical data indicate that the interchange took place toward the left end of both chromosomes. The cytological figures confirm this. Unfortunately it has been impossible to distinguish the insertion regions in the figures, and since the interchange was equal no identification could be made of the specific chromosomes in the interchange complex. No exact measurements have been made of the relative lengths of the interchanged pieces and the normal chromosomes but there are considerable portions in the left ends of the *b-lg* and *pr-$v_2$* chromosomes which are, as yet, unmarked by mutant genes.

*Summary.*—The analysis of a segmental interchange involving the *b-lg* and *pr-$v_2$* chromosomes in Zea is reported. The percentages of crossing-over between *$ts_1$*, *b* and *lg* with the point of interchange in the *b-lg* chromosome are 4, 22 and 47, respectively. The loci of *pr* and *$v_2$* are 18 and 46 crossover units distant from the break in the *pr-$v_2$* chromosome.

Cultures segregating for the genes *b* and *pr* and for the interchanged chromosomes show approximately 31 per cent of crossing-over between *b* and *pr*.

The frequency of double crossovers between *b*, the point of interchange, and *pr* indicate the absence of interference in crossing over.

* Paper No. 180 from the Department of Plant Breeding, Cornell University, Ithaca, N. Y.

† I wish to offer my sincere appreciation to Dr. Barbara McClintock who found semi-sterile-4 in her cultures and generously turned it over for analysis. Dr. McClintock has succeeded (unpublished data) in isolating trisomic races for the *b-lg* and *pr-$v_2$*

chromosomes and has kindly furnished the stocks for the trisomic tests. I also wish to thank Miss Harriet Creighton for her cytological assistance and Dr. C. R. Burnham for some of the stocks used in the experiments.

[1] Belling, John, and A. F. Blakeslee, *Proc. Nat. Acad. Sci.*, **12**, 7–11 (1926).

[2] Brink, R. A., *J. Hered.*, **18**, 266–270 (1927).

[3] Brink, R. A. and Burnham, C. R., *Amer. Nat.*, **63**, 301–316 (1929).

[4] Brink, R. A. *Anat. Rec.* (Abstract), **4**, 280 (1929).

[5] Burnham, C. R., *Proc. Nat. Acad. Sci.*, **16**, 269–277 (1930).

[6] McClintock, Barbara, *Ibid.*, **16**, 791–796 (1930).

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## *MODIFICATION OF MENDELIAN RATIOS IN MAIZE BY MECHANICAL SEPARATION OF GAMETES*

By P. C. Mangelsdorf

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For a number of years the writer, in collaboration with Dr. D. F. Jones and Dr. W. R. Singleton, has been studying the inheritance of a peculiar condition in maize known as "high sugary." Plants of this stock, which are heterozygous for the well-known recessive character, sugary endosperm, produce about 66 per cent of sugary seeds when self-pollinated instead of the expected 25 per cent. When the heterozygote is back-crossed on the recessive, approximately 94 per cent of the seeds, instead of the usual 50 per cent, are sugary. The ability to produce these aberrant ratios is inherited but is transmitted through only about 15 per cent of the ovules and about one per cent of the functional pollen.

All plants which produce high sugary ratios have variable pollen. Measurements of the pollen grains show that the distribution, with respect to length, is bi-modal. Approximately half of the grains are smaller than normal, though quite sound and well packed with reserves, while the remainder are normal. The gene or other condition responsible for the production of tiny pollen is located in the third chromosome, to the left of sugary, and the crossing-over is approximately six per cent. Other characters in this linkage group, including defective endosperm $de_{16}$, tunicate ear *Tu*, tassel seed $Ts_5$, and a newly discovered allelomorph of sugary, are also affected.

We may assume that in "high sugary" plants, which are heterozygous for sugary, the normal pollen grains all carry the gene for sugary except six per cent of crossovers, while the tiny grains all carry starchy genes except the six per cent of crossovers. We may assume further that the tiny pollen grains do not function in competition with normal grains;

hence the percentage of starchy seeds in backcrosses is a direct measure of the crossing-over between normal pollen and sugary endosperm. These assumptions are shown to be reasonably sound by the results of separating the pollen grains according to size by passing the pollen through a series of fine-meshed sieves.

Three sieves, having specified openings of 0.088 mm., 0.074 mm., and 0.062 mm., were used. The fractions remaining in each sieve after shaking, as well as the fraction passing through the finest sieve, were used separately in making pollinations on homozygous sugary plants. Examination of the fractions under the microscope showed that no large, sound pollen grains passed through the finest sieve, though some small grains still remained in the coarsest sieve.

The results from these pollinations, compared with pollinations made with untreated pollen, are set forth in table 1. The experiment was repeated on five different days and the results were essentially the same in each case, hence only the totals and the averages are shown.

TABLE 1

RESULTS OF POLLINATING HOMOZYGOUS SUGARY WITH VARIOUS FRACTIONS OF POLLEN FROM HETEROZYGOUS "HIGH SUGARY" PLANTS

| POLLEN USED | SEEDS PRODUCED | | |
|---|---|---|---|
| | TOTAL | SUGARY | PER CENT SUGARY |
| 1. Not treated | 1170 | 1100 | 94.0 |
| 2. Fraction remaining in sieve 0.088 mm. | 1961 | 1830 | 93.3 |
| 3. Fraction passing through sieve 0.088 mm. but remaining in sieve 0.074 mm. | 2208 | 1915 | 86.7 |
| 4. Fraction passing through sieve 0.074 mm. but remaining in sieve 0.062 mm. | 2566 | 1073 | 41.8 |
| 5. Fraction passing through sieve 0.062 mm | 912 | 236 | 25.9 |

It is noted that the percentages of sugary seeds in Lots 1 and 2 are practically identical, 94.0 and 93.3 per cent, showing that the removal of most of the tiny grains had little or no effect on the gametic ratios and indicating further that only the normal grains ordinarily function.

The next three fractions, represented by the ears in Lots 3, 4, and 5, produced 86.7, 41.8, and 25.9 per cent sugary seeds, respectively. All of these values differ significantly from the percentage in the first fraction, from the untreated pollen, and from each other.

In none of the tests was the percentage of sugary seeds reduced to the six per cent expected if only tiny grains had been included in the fraction. In one of the separate experiments, however, the pollinations made with the lowest fraction averaged 15.3 per cent sugary seeds and one of the ears averaged 8.1 per cent sugary seeds.

The failure to reduce the proportion of sugary seeds to six per cent is probably due to the fact that some of the smaller grains which pass through

the finest sieve are genetically normal and not tiny. Such grains are presumably 94 per cent sugary and six per cent starchy and they would tend to raise the percentage of sugary seeds produced by the pollen fraction in which they are included.

Although the separation of the gametes has not been complete, the evidence seems to be sufficiently conclusive to show that:

1. Normal and tiny pollen grains can be partially separated by simple mechanical means.
2. Separation on the basis of size also results in separation of starchy and sugary genes, furnishing direct evidence that, in this stock, the starchy : sugary genes are genetically associated with differences in size of pollen.
3. The tiny grains, which function only rarely when in competition with normal grains, readily accomplish fertilization when competition with normal grains is reduced or eliminated.

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## *A PROBABLE NEW MUTATION TO WHITE-BELLY IN THE HOUSE MOUSE, MUS MUSCULUS*

By Clyde E. Keeler

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Communicated November 9, 1931

The so-called "agouti" series of allelomorphic genes in the house mouse, as at present understood, is shown in the first two columns of the following table.

| VARIATION | SYMBOLS AS ALLELOMORPHS | SYMBOLS AS LINKED GENES |
|---|---|---|
| 1. Lethal yellow | $A^Y$ | $\widehat{A^Y}$ |
| 2. White-belly agouti | $A^W$ | $\widehat{A^W}$ |
| 3. Agouti (banded hairs) | $A$ | $A$ |
| 4. White-belly non-agouti | $a^W$ | $\widehat{a^W}$ |
| 5. Dark-belly non-agouti | $a$ | $a$ |

Evidence indicates that the second and fourth members of this series are probably not true allelomorphs, but represent members three and five, respectively, in combination with a closely linked dominant gene ($W$) for white-belly. This interpretation is indicated in the last column of the above table.

In 1907 Cuénot discovered the white-belly variation among wild agouti mice in France and the next year Morgan captured white-belly individuals in Massachusetts. Non-agouti white-belly stocks are known to the fancy (Dunn, 1928) in the varieties black-and-tan, blue-and-buff, etc.

Little (1916) reported four mutations to white-belly in agouti animals. Pincus (1929) found a mutation to white-belly in a non-agouti strain maintained at the Bussey Institution. In 1931 Snell reported four probable mutations to white-belly in non-agouti mice from the same laboratory. This gives us a total of twelve times that this gene is thought to have mutated previous to the case now to be placed upon record.

This supposed mutation to white-belly agouti was discovered in experiments designed to test the hypothesis that two distinct loci are involved in the white-belly agouti pheno type. An extensive back-cross had been made (Fig. A) to detect possible crossing-over between the hypothetical *A* and *W* loci. If indisputable crossovers occurred, then independence of the two loci would be proved.

In preparation for the back-cross $F_1$ white-belly agouti mice (Fig. A, 3) were produced by crossing ordinary agoutis (Fig. A, 1) with white-belly non-agoutis (Fig. A, 2), the latter being commonly known as black-and-tans. The $F_1$ white-belly agouti animals were then back-crossed to dark-belly non-agoutis (Fig. A, 4). Two classes of offspring are expected in equal numbers in the case of simple allelomorphism, namely, those represented by 5 and 8 in figure A, these being identical in character with the grandparents, 1 and 2, figure A. But if a combination occurred such as is shown in 6 or 7, figure A, it would prove the distinctness of the loci of the genes for agouti and for white-belly, the only possible alternative being that a fresh mutation had occurred either from ordinary agouti to white-belly agouti or from white-belly non-agouti to dark-belly non-agouti.

In certain litters of the back-cross experiments both classes of apparent crossovers have been observed, viz., 13 dark-belly non-agouti individuals and 19 white-belly agoutis. These mice were all sired by the same $F_1$ white-belly male. Obviously there is something peculiar about his genetic constitution, since in a total of 32 offspring he produces *only* crossover individuals, whereas these are expected not oftener than once among 500 individuals, as indicated by other unpublished observations.

Accordingly, in the case of this male (Fig. B, 3), white-belly and agouti must have been transmitted in the same chromosome, for his progeny by ordinary non-agouti mates (Fig. B, 4) are of the classes shown in B, 5 and 6.

But from his pedigree (Fig. B, 1 and 2) we should expect *A* and *W* to show repulsion, that is, to be borne in different chromosomes, since *A* was borne only by the agouti parent (Fig. B, 1) and *W* only by the black-and-tan parent (Fig. B, 2). A crossover in either of the parents would

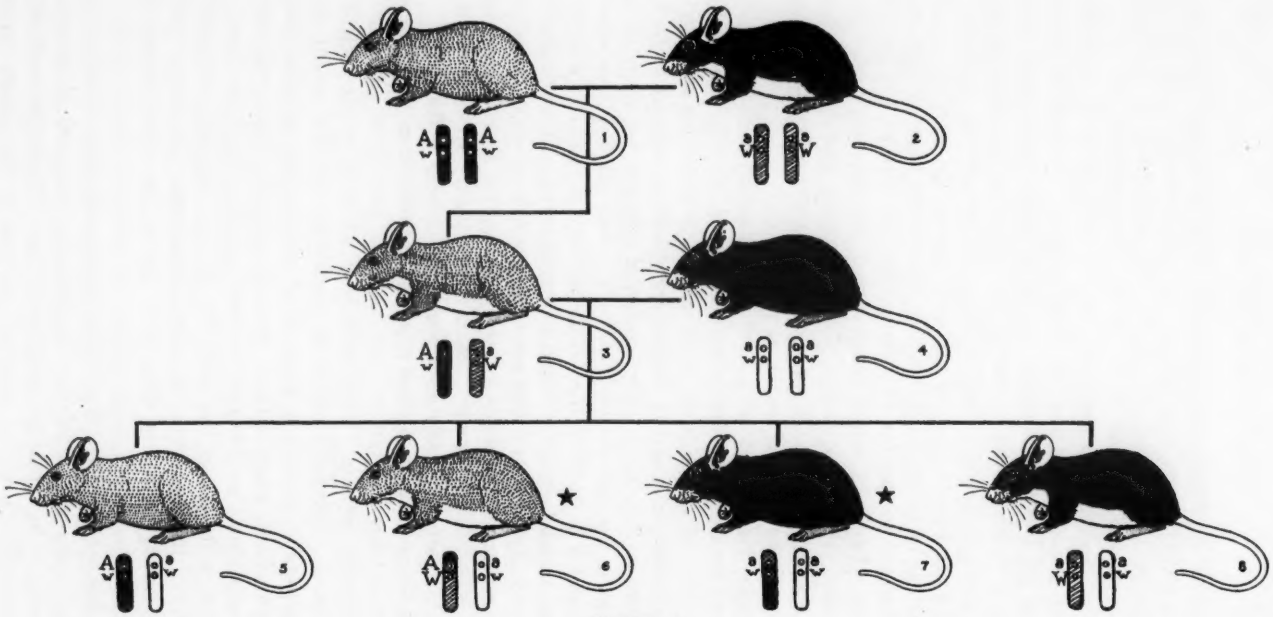


FIGURE A

Diagram representing the crosses used to detect possible independence of the agouti gene ($A$) and the white-belly gene ($W$). Their normal allelomorphs are non-agouti ($a$) and dark-belly ($w$), respectively. The loci in question are represented as circles within a pair of homologous chromosomes. Unstarred $F_2$ classes (5, 8) represent types due to simple recombination of chromosomes. Starred $F_2$ classes (6, 7) represent types expected from crossover gametes occurring in the white-belly agouti $F_1$ individuals (3).

not have resulted in the production of a chromosome bearing both *A* and *W*. Such a crossover in the fertilized egg itself from which this male developed would account for his peculiarity but there is no ground for assuming such an unheard-of occurrence. Hence it seems probable that

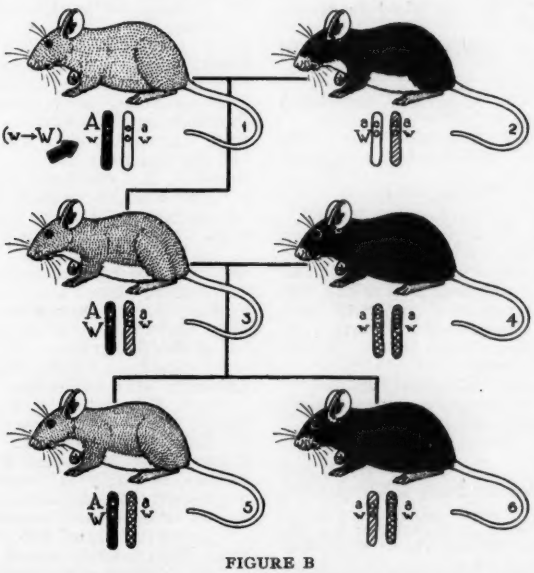


FIGURE B

Diagram illustrating the probable origin of the new white-belly mutation. Symbols employed are the same as in figure A.

a mutation from *w* to *W* occurred in the parental gamete furnished by the agouti parent (Fig. B, 1) or what seems less likely a reverse mutation* from *a* to *A* in the parental gamete furnished by the black-and-tan parent (Fig. B, 2).

* Only one reverse mutation has ever been reported in the mouse (Keeler, 1931).

## LITERATURE CITED

Cuénot, L., *Archv. Zoöl. Exp. et Gen.*, **4**, 1907.
Dunn, L. C., *Proc. Nat. Acad. Sci.*, **14**, 816–819, 1928.
Keeler, C. E., *Ibid.*, **17**, 497–499, 1931.
Little, C. C., *Amer. Nat.*, **50**, 335–349, 1926.
Pincus, G., *Proc. Nat. Acad. Sci.*, **15**, 85–88, 1929.
Snell, G. D., *Genetics*, **16**, 42–72, 1931.

# INDEX


Abelian groups (Miller), 320.
Abelian integrals (Hodge), 643.
Abilities, human (Miles), 627.
Acetyl choline, effect on chromatophores (Parker), 596.
ACKERMAN, J. W. See Bancroft, W. D., 407, 480.
Activation of charcoal (Bancroft and Whearty), 181.
Adrenalin (Sears), 280.
"Afterglow" in Oenothera Rubricalyx (Cleland), 437.
Air resistance of projectiles (Epstein), 532.
ALBERT, A. A. Normal Division Algebras of Order $2^{2m}$, 389.
Alcohol, methyl (Rossini), 343.
Algebras, normal division (Albert), 389.
Allelomorphs of drosophila (Sturtevant and Schultz), 265.
Ampullar organs of fish ears (Bowen), 192.
Arithmetic of polynomials (Carlitz), 120.
Aromatic substitution products with fluorine (Bancroft and Whearty), 183.
Atomic collisions (Rice), 34.
Atropine (Cushing), 178.
Automorphisms (Miller), 39, 320.
Axioms for differential geometry (Veblen and Whitehead), 551.

BALL, E. G., AND CLARK, W. M. A Potentiometric Study of Epinephrine, 347.
BANCROFT, W. D., AND ACKERMAN, J. W. The Solid Solution Theory of Dyeing, 480.
BANCROFT, W. D., ACKERMAN, J. W., AND GALLAGHER, C. Optical Sensitization, 407.
BANCROFT, W. D., GUTSELL, R. S., AND RUTZLER, J. E., JR. Reversible Coagulation in Living Tissue. IX, 637.
BANCROFT, W. D., AND RICHTER, G. H. Reversible Coagulation in Living Tissue. III, 294; IV, 410.
BANCROFT, W. D., AND RUTZLER, JR., J. E. Reversible Coagulation in Living Tissue. I, 105; II, 186; V, 482; VI, 570; VII, 575; VIII, 597.
BANCROFT, W. D., AND WHEARTY, JR., S. F. Activation of Charcoal, 181.
——. Aromatic Substitution Products with Fluorine, 183.
BATEMAN, H. Solutions of a Certain Partial Differential Equation, 562.
——. Relations between Confluent Hypergeometric Functions, 689.
BAUER, P. S. Dissipative Dynamical Systems. I, 311.
BERRY, A. C. A Metric for the Space of Measurable Functions, 456.
BENEDICT, F. G., AND FOX, E. L. Body Temperature and Heat Regulation of Large Snakes, 584.
Binary quadratic forms (Pall), 368.
BIRKHOFF, G. D. Proof of a Recurrence Theorem for Strongly Transitive Systems, 650.
——. Proof of the Ergodic Theorem, 656.
BLESS, A. A. The Composition of the Interior of the Earth, 225.
Body pattern as related to mammary gland secretion (Gowen), 518.
Body temperature of large snakes (Benedict and Fox), 584.
BORSOOK, H., AND WINEGARDEN, H. M. The Work of the Kidney in the Production of Urine, 3.
——. The Energy Cost of the Excretion of Urine, 13.
——. On the Specific Dynamic Action of Protein, 75.
BOWEN, R. E. Movement of the So-Called Hairs in the Ampullar Organs of Fish Ears, 192.
——. The Cupula of the Ear, 413.
BRAMLEY, A. Radioactive Disintegration, 579.
BREIT, G. On the Interpretation of Dirac's $\alpha$-Matrices, 70.
BRINK, R. A. See Cooper, D. C., 334.
BROOKS, M. M. The Penetration of 1-Naphthol-2-Sulphonate Indophenol, *o*-Chloro Phenol Indophenol and *o*-Cresol Indophenol into Valonia, 1.
BULL, H. B., AND GORTNER, R. A. Studies on Electrokinetic Potentials. IX. The Electrical Field of Force at Liquid-liquid Interfaces, 288.
BYERLY, P. The California Earthquakes of Nov. 28, 1929, and the Surface Layers of the Earth in California, 91.

Calcium metaborate (Zachariasen), 617.
Calculus of variations (La Paz), 459.
California earthquakes of Nov. 28, 1929 (Byerly), 91.
Canal rays, high speed (Lawrence and Sloan), 64.
Carbon dioxide, Raman spectrum (Houston and Lewis), 229.
CARLITZ, L. The Arithmetic of Polynomials in a Galois Field, 120.
CASSEN, B. Electromagnetic Fields Derived from Non-commuting Potentials, 430.
Catfish (Parker and Paine), 594.
Cerebral ventricles (Cushing), 163, 171, 178, 239, 248.
Charcoal, activation of (Bancroft and Whearty), 181.
Chi-square (Wilson and Hilferty), 684.

Chromatophores of crustaceans (Perkins and Snook), 282.
Chromatophores, effects of acetyl choline (Parker), 596.
Chromosomes in maize (Cooper and Brink), 334.
Chromosomes, in Sciara (Metz and Smith), 195; (Metz and Schmuck), 272.
Chromosomes in Zea Mays (McClintock), 485; (Creighton and McClintock), 492.
Chromosomes of lathyrus tuberosus (Fisk), 511.
Cilia of naphthys bucera (Coonfield), 416.
CLARK, W. M. See Ball, E. G., 347.
CLELAND, R. E. The Probable Origin of Oenothera Rubricalyx "Afterglow" on the Basis of the Segmental Interchange Theory, 437.
CLEMENTS, F. The Correlation between Tanning and Unexposed Skin Color as Recorded by the Color Top, 285.
Coagulation, reversible, in living tissue (Bancroft and Rutzler), 105, 186, 482, 570, 575, 597; (Bancroft and Richter), 294, 410; (Bancroft, Gutsell and Rutzler), 637.
COBLENTZ, W. W., STAIR, R., AND HOGUE, J. M. The Spectral Erythemic Reaction of the Human Skin to Ultra-Violet Radiation, 401.
Collision of atoms (Rice), 34.
Color changes in the sea-urchin arbacia (Parker), 592.
Coloring of graphs (Whitney), 122.
Conjugate base (Morse), 319.
Continuity, equation of (Lindsay), 420.
COONFIELD, B. R. The Cilia of Nephthys Bucera, 416.
COOPER, D. C., and BRINK, R. A. Cytological Evidence for Segmental Interchange between Non-Homologous Chromosomes in Maize, 334.
Coördinates, local (Michal), 132.
Cosmic rays (Störmer), 62, (Epstein), 160.
Crayfish, giant fibres in (Warner), 140.
CREIGHTON, H. B., AND MCCLINTOCK, B. A Correlation of Cytological and Genetical Crossing-Over in Zea Mays, 492.
Crossing-over (Morgan), 270.
Crossing-over and mutation (Sax), 601.
Crossing-over in Zea Mays (Creighton and McClintock), 492.
Crustaceans, chromatophores of (Perkins and Snook), 282.
Crystal lattice of calcium metaborate (Zachariasen), 617.
Crystals (Zwicky), 524.
Cupula of the ear (Bowen), 413.
Curvature, multivectorial (Knebelman), 43.
CUSHING, H. I. The Reaction to Posterior Pituitary Extract (Pituitrin) When Introduced into the Cerebral Ventricles, 163.
——. II. The Similarity in the Response to Posterior Lobe Extract (Pituitrin) and to Pilocarpine When Injected into the Cerebral Ventricles, 171.
——. III. The Action of Atropine in Counteracting the Effects of Pituitrin and of Pilocarpine Injected into the Cerebral Ventricles, 178.
——. IV. The Method of Action of Pituitrin Introduced into the Ventricle, 239.
——. V. The Counteractive Effect of Tribomethanol (Avertin) on the Stimulatory Response to Pituitrin Injected in the Ventricle, 248.
——. VI. Concerning a Possible "Parasympathetic Center" in the Diencephalon, 253.

DALLA VALLE, J. M. Note on the Heaviside Expansion Formula, 678.
Dermal melanophores (Field), 137.
Diatomic gases (Wollan), 475.
Diencephalon (Cushing), 253.
Differential equations (Ritt), 366; (Bateman), 562.
Differential geometry (Veblen and Whitehead), 551.
Dirac's $\alpha$-Matrices (Breit), 70.
Disintegration, radioactive (Bramley), 579.
Dissipative dynamical systems (Bauer), 311.
Division algebras (Albert), 389.
DOBZHANSKY, T., AND SCHULTZ, J. Evidence for Multiple Sex Factors in the X-Chromosome of Drosophila Melanogaster, 513.
DOUGLAS, J. The Least Area Property of the Minimal Surface Determined by an Arbitrary Jordan Contour, 211.
Drosophila, allelomorphs (Sturtevant and Schultz), 265.
Drosophila, crossing-over (Morgan), 270.
Drosophila melanogaster (Dobzhansky and Schultz), 513.
DRYDEN, A. L., JR. Accuracy in Percentage Representation of Heavy Mineral Frequencies, 233.
Dynamic action of protein (Borsook and Winegarden), 75.
Dynamical trajectories (Kasner), 370.
Dynamical systems (Bauer), 311.
Dyeing, solid solution theory of (Bancroft and Ackerman), 480.

Ear, cupula of (Bowen), 413.
Ears of fishes (Bowen), 192.
Earth, interior of (Bless), 225.

Earthquakes, California, of Nov. 28, 1929 (Byerly), 91.
EAST, E. M. Immunity to Sugar Cane Mosaic Acquired by the Host, 331.
Einstein universe (Tolman), 153.
Electric conductivity in metals (Hall), 392, 427.
Electric units (Kennelly), 147.
Electrokinetic potentials (Bull and Gortner), 288.
Electromagnetic fields (Cassen), 430.
Electrons, free (Kennard), 58.
Electrostatic analog of gravitational red shift (Kennedy and Thorndike), 620.
Epinephrine, potentiometric study of (Ball and Clark), 347.
EPSTEIN, P. S. Answer to Prof. Störmer's Remark, 160.
——. On the Air Resistance of Projectiles, 532.
Equations, algebraic differential (Ritt), 366.
Equation of continuity (Lindsay), 420.
Ergodic theorem (Birkhoff), 656.
Erythemic reaction of human skin (Coblentz, Stair and Hogue), 401.
Euler equations (LaPaz), 459.

Fermat's last theorem (Vandiver), 661.
FIELD, E. H. Reactions of Dermal Melanophores in Necturus to Heat and to Cold, 137.
Field, Galois (Carlitz), 120.
Field theory, unified (Thomas), 48, 111, 199, 325.
Fields, electromagnetic (Cassen), 430.
FISK, E. L. The Chromosomes of Lathyrus Tuberosus, 511.
Fluid, viscous (MacCreadie), 381.
Fluorine (Bancroft and Whearty), 183.
Fourier series (Hille and Tamarkin), 376.
Fowl, domestic (Waters), 440.
FOX, E. L. See Benedict, F. G., 584.
Frequencies, mineral (Dryden), 233.
Frogs, Melanophores in (Sears), 280.
FROST, H. B. Uncomplicated Trisomic Inheritance of Purple Versus Red in *Matthiola Incana*, 434.
——. Trisomic Inheritance of Doubleness, Complicated by Lethals, in *Matthiola Incana*, 499.
Functional psychoses (Lang and Paterson), 603.
Functions, hypergeometric (Bateman), 689.
Function space (Berry), 456.
Function space-time manifolds (Michal), 217.

GALLAGHER, C. See Bancroft, W. D., 407.
Galois field (Carlitz), 120.
Gases, diatomic (Wollan), 475.
Gases, X-ray scattering in (Woo), 467, 470.
Geometry, differential (Veblen and Whitehead), 551.
GORDON, M. The Hereditary Basis for Melanosis in Hybrids of Mexican Killifishes, 276.
GORTNER, R. A. See Bull, H. B., 288.
GOWEN, J. W. Body Pattern as Related to Mammary Gland Secretion, 518.
Graphs (Whitney), 122, 125.
Gravitational red shift (Kennedy and Thorndike), 620.
Group manifolds (Smith), 674.
Groups, Abelian (Miller), 320.
Groups (Miller), 39, 000; (Kiang), 142.
GRUNDFEST, H. The Relative Effectiveness of Spectral Radiation for the Vision of the Sun-Fish, Lepomis, 359.
GUTSELL, R. S. See Bancroft, W. D., 637.

Hairs in fish ears (Bowen), 192.
Hadamard, theorems of (Trijitzinsky), 568.
HALL, E. H. Electric Conductivity and Optical Absorption in Metals, Once More, 392; (Supplementary), 427.
Hamiltonian systems (Koopman), 315.
Heat of combustion of methyl alcohol (Rossini), 343.
Heat reactions of dermal melanophores (Field), 137.
Heat regulation of large snakes (Benedict and Fox), 584.
Heating of liquids by absorption of sound (Richards), 611.
Heaviside expansion formula (Dalla Valle), 678.
High-frequency sound waves (Richards), 611.
High vacuum pressure control apparatus (Ho), 548.
Hilbert space (Koopman), 315.
HILFERTY, M. M. See Wilson, E. B., 684.
HILLE, E., AND TAMARKIN, J. D. On the Summability of Fourier Series. Fourth Note, 376.
Histological polarization in crayfish (Warner), 140.
History of mathematics (Miller), 463.
HO, T. L. High Vacuum Pressure Control Apparatus, 548.
HODGE, W. V. D. Further Properties of Abelian Integrals Attached to Algebraic Varieties, 643.
HODGES, J. H., AND LINHORST, E. F. The Thermal Decomposition of Nitrogen Pentoxide at Low Pressures, 28.
HOGUE, J. M. See Coblentz, W. W., 401.
House mouse, pigmentation in (Keeler), 497.
HOUSTON, W. V., AND LEWIS, C. M. Rotational Raman Spectrum of $CO_2$, 229.

HOVGAARD, W. The Distribution of Stresses in Welded and Riveted Connections, 351.
Human abilities and the life span (Miles), 627.
Human skin (Coblentz, Stair and Hogue), 401.
Hurwitz, theorems of (Trijitzinsky), 568.
Hydrographic surveying (Shalowitz), 445.
Hypergeometric functions (Bateman), 689.

Immunity in sugar cane (East), 331.
Inheritance in *Matthiola Incana* (Frost), 434.
Inheritance of weight in fowl (Waters), 440.
Inheritance, trisomic (Frost), 499.
Integrals, Abelian (Hodge), 643.
Interfaces, liquid-liquid (Bull and Gortner), 288.
Inverse commutator subgroups (Miller), 144.

Jordon contour (Douglas), 211.

KASNER, E. Dynamical Trajectories and the $\infty^3$ Plane Sections of a Surface, 370.
KEELER, C. E. The Independence of Dominant Spotting and Recessive Spotting ("Piebald") in the House Mouse, 101.
——. A Reverse Mutation from "Dilute" to "Intense" Pigmentation in the House Mouse, 497.
——. A Probable New Mutation to White-Belly in the House Mouse, Mus Musculus, 700.
KENNARD, E. H. Quantum-Mechanical Motion of Free Electrons in Electromagnetic Fields, 58.
KENNEDY, R. J., AND THORNDIKE, E. M. A Search for an Electrostatic Analog to the Gravitational Red Shift, 620.
KENNELLY, A. E. The Convention of Equidimensional Electric and Magnetic Units, 147.
KIANG, T.-H. On the Groups of Orientable Two-Manifolds, 142.
Kidneys, work of (Borsook and Winegarden), 3, 13.
Killifishes, Mexican (Gordon), 276.
KNEBELMAN, M. S. Multivectorial Curvature, 43.
KOOPMAN, B. O. Hamiltonian Systems and Transformations in Hilbert Space, 315.

LANG, H. B., AND PATERSON, J. A. A Preliminary Report on Functional Psychoses, 603.
LA PAZ, L. The Euler Equations of Problems of the Calculus of Variations with Prescribed Transversality Conditions, 459.
Lathyrus tuberosus (Fisk), 511.
Lattice, crystal (Zachariasen), 617.
LAWRENCE, E. O., AND SLOAN, D. H. The Production of High Speed Canal Rays without the Use of High Voltages, 64.
LEWIS, C. M. See Houston, W. V., 229.
LINDSAY, R. B. Wave Motion and the Equation of Continuity, 420.
LINHORST, E. F. See Hodges, J. H., 28.
Linkage values in zea (Rhoades), 694.
Liquid-liquid interfaces (Bull and Gortner), 288.

MACCREADIE, W. Y. On the Stability of the Motion of a Viscous Fluid, 381.
MCCLINTOCK, B. The Order of the Genes C, Sh and Wx in Zea Mays with Reference to a Cytologically Known Point in the Chromosome, 485.
——. See Creighton, H. B., 492.
MACCURDY, G. G. The Use of Rock Crystal by Paleolithic Man, 633.
Maize (Mangelsdorf), 698.
Maize, non-homologous chromosomes in (Cooper and Brink), 334.
Man, paleolithic (MacCurdy), 633.
MANGELSDORF, P. C. Modification of Mendelian Ratios in Maize by Mechanical Separation of Gametes, 698.
Manifolds (Kiang), 142; (Michal), 217.
Manifolds, group (Smith), 674.
Mathematics, history of (Miller), 463.
Matrices (Murnaghan and Wintner), 417, 676.
Matrices, Dirac's (Breit), 70.
Matthiola incana (Frost), 434, 499.
Maupertuis, principle of (Murnaghan), 128.
Measurable functions (Berry), 456.
Melanophores, dermal (Field), 137.
Melanophores in the frog (Sears), 280.
Melanosis in hybrids of Mexican killifishes (Gordon), 276.
Mendelian ratios in maize (Mangelsdorf), 698.
Metals (Hall), 392, 427.
Metric for function space (Berry), 456.
METZ, C. W., AND SMITH, H. B. Further Observations on the Nature of the X-Prime (X′) Chromosome in Sciara, 195.
METZ, C. W., AND SCHMUCK, M. L. Differences between Chromosome Groups of Soma and Germ-Line in Sciara, 272.
Mexican killifishes (Gordon), 276.
Mice, yellow fever in (Sellards), 339.
MICHAL, A. D. Notes on Scalar Extensions of Tensors and Properties of Local Coördinates, 132.

——. Function Space-Time Manifolds, 217.
MILES, W. R. Measures of Certain Human Abilities throughout the Life Span, 627.
MILLER, G. A. Groups Which Admit Five-Eighths Automorphisms, 39.
——. Inverse Commutator Subgroups, 144.
——. Automorphisms of Order 2 of an Abelian Group, 320.
——. Theorems Relating to the History of Mathematics, 463.
——. Groups Involving a Small Number of Conjugates, 691.
Mineral frequencies (Dryden), 233.
Minimal surface (Douglas), 211.
Monkeys, yellow fever in (Sellards), 339.
MORGAN, L. V. Proof That Bar Changes to Not-Bar by Unequal Crossing-Over, 270.
MORSE, M. The Order of Vanishing of the Determinant of a Conjugate Base, 319.
Mosaic, sugar cane (East), 331.
Mouse (Keeler), 101, 497, 700.
MURNAGHAN, F. D. The Principle of Maupertuis, 128.
——. See Wintner, A,. 676.
MURNAGHAN, F. D., AND WINTNER, A. A Canonical Form for Real Matrices under Orthogonal Transformations, 417.
Music (Seashore), 623.
Mutation and crossing-over (Sax), 601.
Mutation in house mouse (Keeler), 700.

Nephthys bucera (Coonfield), 416.
Nerve, lateral-line (Parker and Paine), 594.
Nerve-fibres of taste-buds (Torrey), 589.
Nitrogen pentoxide at low pressures (Hodges and Linhorst), 28.
Nitrous oxide (Ramsperger and Waddington), 103.
Nitrous oxide, decomposition of (Ramsperger and Waddington), 103.

Oenothera rubricalyx "Afterglow" (Cleland), 437.
Optical absorption in metals (Hall), 392, 427.
Optical sensitization (Bancroft, Ackerman and Gallagher), 407.
Orientable two-manifolds (Kiang), 142.
Orthogonal transformations (Murnaghan and Wintner), 417.

PAINE, V. L. See Parker, G. H., 594.
Paleolithic man (MacCurdy), 633.
PALL, G. The Number of Representations Function for Positive Binary Quadratic Forms, 368.
Parasympathetic center (Cushing), 253.
PARKER, G. H. The Color Changes in the Sea-Urchin Arbacia, 592.
——. Effects of Acetyl Choline on Chromatophores, 596.
PARKER, G. H., AND PAINE, V. L. Progressive Degeneration of the Lateral-Line Nerve in the Catfish, 594.
Partial differential equation (Bateman), 562.
PATERSON, J. A. See Lang, H. B., 603.
Penetrability of Valonia (Brooks), 1.
PERKINS, E. B., AND SNOOK, T. Control of Pigment Migration in the Chromatophores of Crustaceans, 282.
Pigment migration (Perkins and Snook), 282.
Pigmentation in house mouse (Keeler), 497.
Pilocarpine (Cushing), 171.
Pituitrin (Cushing), 163, 171, 178, 239, 248; (Sears), 280.
Polarization, histological, in crayfish (Warner), 140.
Polynomials, arithmetic of (Carlitz), 120.
Posterior pituitary extract (Cushing), 163.
Potentials, electrokinetic (Bull and Gortner), 288.
Potentials, non-commuting (Cassen), 430.
Pressure control apparatus (Ho), 548.
Projectiles, air resistance of (Epstein), 532.
Protein, specific dynamic action of (Borsook and Winegarden), 75.
Psychoses, functional (Lang and Paterson), 603.

Quadratic forms (Pall), 368.
Quantum mechanical motion (Kennard), 58.
Quantum theories (Seeger), 301.

Radioactive disintegration (Bramley), 579.
Radiation, ultra-violet (Coblentz, Stair and Hogue), 401.
Raman spectrum $CO_2$ (Houston and Lewis), 229.
RAMSPERGER, H. C., AND WADDINGTON, G. The Interpretation of the Thermal Decomposition of Nitrous Oxide, 103.
Recurrence theorem (Birkhoff), 650.
Reversible coagulation in living tissue (Bancroft and Rutzler), 105, 186, 482, 570, 575, 597; (Bancroft and Richter), 294, 410; (Bancroft, Gutsell and Rutzler), 637.
RHOADES, M. M. Linkage Values in an Interchange Complex in Zea, 694.
RICE, O. K. On the Transfer of Energy between Atoms at Collision, 34.
RICHARDS, W. T. The Heating of Liquids by the Absorption of Sound, and Its Relation to the Energy of Intense High-Frequency Sound Waves 611.
RICHTER, G. H. See Bancroft, W. D., 294, 410.

RITT, J. F. Systems of Algebraic Differential Equations, 366.
ROSSINI, F. D. The Heat of Combustion of Methyl Alcohol, 343.
RUTZLER, J. E., JR. See Bancroft, W. D., 105, 186, 482, 570, 575, 597, 637.

SAX, K. Crossing-Over and Mutation, 601.
Scattering by diatomic gases (Wollan), 475.
Scattering of X-rays by gases (Woo), 467, 470.
SCHULTZ, J. See Dobzhansky, T., 513.
——. See Sturtevant, A. H., 265.
Sciara, chromosomes in (Metz and Smith), 195; (Metz and Schmuck), 272.
SCHMUCK, M. L. See Metz, C. W., 272.
SEARS, M. The Responses of the Deep-Seated Melanophores in the Frog to Adrenalin and Pituitrin, 280.
SEASHORE, C. E. The Natural History of the Vibrato, 623.
Sea-urchin arbacia (Parker), 592.
Secretion of mammary gland (Gowen), 518.
SEEGER, R. J. A Critique of Recent Quantum Theories, 301.
SELLARDS, A. W. The Behavior of the Virus of Yellow Fever in Monkeys and Mice, 339.
Sensitization, optical (Bancroft, Ackerman and Gallagher), 407.
Sex factors in Drosophila (Dobzhansky and Schultz), 513.
SHALOWITZ, A. L. The Physical Basis of Modern Hydrographic Surveying, 445.
Shift, gravitational red (Kennedy and Thorndike), 620.
Singularities, composition of (Trjitzinsky), 568.
Skin color (Clements), 285.
Skin, human (Coblentz, Stair and Hogue), 401.
SLOAN, D. H. See Lawrence, E. O., 64.
SMITH, H. B. See Metz, C. W., 195.
SMITH, P. A. Properties of Group Manifolds, 674.
Snakes, temperature of (Benedict and Fox), 584.
SNOOK, T. See Perkins, E. B., 282.
Solid solution theory of dyeing (Bancroft and Ackerman), 480.
Sound waves, high frequency (Richards), 611.
Spaces, Hilbert (Koopman), 315.
Spectrum of $CO_2$ (Houston and Lewis), 229.
Spotting, in house mouse (Keeler), 101.
Stability of motion of a viscous fluid (MacCreadie), 381.
STAIR, R. See Coblentz, W. W., 401.
STÖRMER, C. Remarks on a Paper: Note on the Nature of Cosmic Rays, by Paul S. Epstein, 62.
Stresses in welded and riveted connections (Hovgaard), 351.
STURTEVANT, A. H., AND SCHULTZ, J. The Inadequacy of the Sub-Gene Hypothesis of the Nature of the Scute Allelomorphs of Drosophila, 265.
Sub-gene theory (Sturtevant and Schultz), 265.
Subgroups, inverse commutator (Miller), 144.
Sugar cane mosaic, immunity (East), 331.
Summability of Fourier Series (Hille and Tamarkin), 376.
Sun-fish, Lepomis (Grundfest), 359.
Surface, minimal (Douglas), 211.
Surfaces, plane sections of (Kasner), 370.
Surveying, hydrographic (Shalowitz), 445.

Tanning of skin (Clements), 285.
Taste-buds and their nerve-fibres (Torrey), 589.
Tensors, scalar extensions of (Michal), 132.
Thermal decomposition of nitrogen pentoxide (Hodges and Linhorst), 28.
Thermal decomposition of nitrous oxide (Ramsperger and Waddington), 103.
Thermodynamic equilibrium in universe (Tolman), 153.
THOMAS, T. Y. On the Unified Field Theory. III, 48; IV, 111; V, 199, VI, 325.
THORNDIKE, E. M. See Kennedy, R. J., 620.
Tissue, coagulation in (Bancroft and Rutzler), 105, 186, 482, 570, 575, 597; (Bancroft and Richter), 294, 410; (Bancroft, Gutsell and Rutzler), 637.
TOLMAN, R. C. On Thermodynamic Equilibrium in a Static Einstein Universe, 153.
TORREY, T. W. The Relation of Taste-buds to Their Nerve-Fibres, 589.
Trajectories, dynamical (Kasner), 370.
Transformations (Koopman), 315.
Transformations, orthogonal (Murnaghan and Wintner), 417.
Transversality conditions of calculus of variations (La Paz), 459.
Tribomethanol (avertin) (Cushing), 248.
TRJITZINSKY, W. J. A Synthesis of the Theorems of Hadamard and Hurwitz on Composition of Singularities, 568.

Ultra-violet radiation (Coblentz, Stair and Hogue), 401.
Unified field theory (Thomas), 48, 111, 199, 325.
Units, magnetic (Kennelly), 147.
Universe, static Einstein (Tolman), 153.
Urine, work in production of (Borsook and Winegarden), 3, 13.

Valonia, penetrability of (Brooks), 1.
VANDIVER, H. S. Summary of Results and Proofs on Fermat's Last Theorem (Sixth Paper), 661.
VEBLEN, O., AND WHITEHEAD, J. H. C. A Set of Axioms for Differential Geometry, 551, 660.
Ventricles, cerebral (Cushing), 163, 171, 178, 239, 248.
Vibrato (Seashore), 623.
Virus of yellow fever (Sellards), 339.
Viscous fluid (MacCreadie), 381.
Vision of sun-fish (Grundfest), 359.

WADDINGTON, G. See Ramsperger, H. C., 103.
WARNER, S. G. Histological Polarization of Lateral Giant-Fibres in the Crayfish, 140.
Wave motion (Lindsay), 420.
WATERS, N. F. Inheritance of Body Weight in Domestic Fowl, 440.
Weight, inheritance in fowl (Waters), 440.
Welded and riveted connections (Hovgaard), 351.
WHEARTY, JR., S. F. See Bancroft, W. D., 181, 183.
WHITEHEAD, J. H. C. See Veblen, O., 551, 660.
WHITNEY, H. The Coloring of Graphs, 122.
——. Non-Separable and Planar Graphs, 125.
WILSON, E. B., AND HILFERTY, M. M. The Distribution of Chi-Square, 684.
WINEGARDEN, H. M. See Borsook, H., 3, 13, 75.
WINTNER, A. See Murnaghan, F. D., 417.
WINTNER, A., AND MURNAGHAN, F. D. On a Polar Representation of Non-Singular Square Matrices, 676.
WOLLAN, E. O. Note on Scattering by Diatomic Gases, 475.
WOO, Y. H. On the Intensity of Total Scattering of X-Rays by Gases. I, 467; II, 470.

X-Chromosome of drosophila (Dobzhansky and Schultz), 513.
X-prime chromosome in sciara (Metz and Smith), 195.
X-rays, scattering of (Woo), 467, 470.

Yellow fever (Sellards), 339.

ZACHARIASEN, W. H. The Crystal Lattice of Calcium Metaborate, $CaB_2O_4$, 617.
Zea (Rhoades), 694.
Zea mays (McClintock), 485; (Creighton and McClintock), 492.
ZWICKY, F. Why Crystals Exist, 524.